

r-Regularity

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Abstract We provide a characterization of r -regular sets in terms of the Lipschitz regularity of normal vector fields to the boundary.

Keywords r -regularity · normal vector field · C^1 -boundary · Lipschitz projection · Euclidean distance

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1 Introduction

The fundamental task of digital image processing is to recognize the properties of real objects given their digital images, i.e., discrete data generated by some image acquisition system. An important question for devising reliable image analysis algorithms is: *which shapes can be digitized without changes in the fundamental geometric or topological properties?* Most of the known answers to this question consider, as suitable shapes, the class of r -regular sets (see [8,9,11,12,15–17]). Furthermore, r -regular sets have been applied in the context of surface reconstruction and image segmentation [10,14,18], which shows that the topic of this paper (r -regularity) is very important in the context of many applications. These applications also include the authors' motivation which comes from the study of smooth non-deterministic dynamical systems, i.e., the

dynamics of 'smooth' point-set maps on a compact manifold, where r -regular sets can appear as dynamically invariant sets (see [5]).

In [9], conditions were derived relating properties of regular sets to the grid size of the sampling device which guarantee that a regular object and its digital image are topologically equivalent. To obtain the topological equivalence the authors used the fact that a regular set is always bounded by a codimension one manifold. This property was conjectured in [9, p. 145] and was proved recently by the authors in [4]. One of the reviewers of the work in [4] gave us a suggestion concerning future work, namely a characterization of r -regular sets in terms of the Lipschitz regularity of normal vector fields to the boundary. In this paper we provide this characterization.

2 Preliminary Definitions

In this section we summarize the basic smoothness concepts used in the proofs. For more details, we refer the reader to [13]. Let $\mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$ denote the vector space of linear maps $L : \mathbb{R}^n \rightarrow \mathbb{R}^m$. Given some open set $U \subseteq \mathbb{R}^n$ in the Euclidean space \mathbb{R}^n , a map $f : U \rightarrow \mathbb{R}^m$ is said to be of class C^1 if and only if there is a continuous function $Df : U \rightarrow \mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$ such that for $x \in U$, as $v \rightarrow 0$ in \mathbb{R}^n , one has $f(x+v) = f(x) + Df_x(v) + o(\|v\|)$, i.e., given $x \in U$ and $\varepsilon > 0$ there is $\delta > 0$ such that for every $v \in \mathbb{R}^n$ with $\|v\| \leq \delta$ and $x + v \in U$ one has $\|f(x+v) - f(x) - Df_x(v)\| \leq \varepsilon \|v\|$. For each $x \in U$, the linear map Df_x is unique and called the *differential* of f at point x . The map $Df : U \rightarrow \mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$ is called the *total derivative* of f . A map $f : U \rightarrow \mathbb{R}^m$ is said to be of class $C^{1+\text{Lip}}$ if and only if f is of class C^1 and

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$Df : U \rightarrow \mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$ is Lipschitz. We shall denote by $\text{Lip}(f)$ the Lipschitz constant of a Lipschitz map f .

Rademacher's theorem states that a Lipschitz map is differentiable almost everywhere (see [6, Section 3.1.6.]).

Theorem 1 (*Rademacher's theorem*) *If $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is Lipschitz, then f is differentiable at almost all points of \mathbb{R}^n .*

This work deals essentially with class $C^{1+\text{Lip}}$ differentiability, but since we mention C^2 smoothness we also provide the definition. A map $f : U \rightarrow \mathbb{R}^m$ is said to be of class C^2 if and only if f is of class C^1 and $Df : U \rightarrow \mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$ is also of class C^1 . The total derivative $D(Df)$ is a function with values in the vector space $\mathcal{L}(\mathbb{R}^n, \mathcal{L}(\mathbb{R}^n, \mathbb{R}^m))$, which can be identified with the space of symmetric bilinear forms $B : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^m$. Hence, $D(Df)_x$ identifies with a symmetric bilinear form $D^2f_x : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^m$, referred as the second order differential of f at point x . It follows from the definition of a class C^2 map that for any $x \in U$, $f(x+v) = f(x) + Df_x(v) + \frac{1}{2}D^2f_x(v, v) + o(\|v\|^2)$, as $v \rightarrow 0$ in \mathbb{R}^n . Equivalently, a function is of class C^2 if and only if it has continuous partial derivatives up to the second order.

Given open sets $U, V \subseteq \mathbb{R}^n$, a map $h : U \rightarrow V$ is called a *diffeomorphism* of class C^k if and only if $h : U \rightarrow V$ is bijective, while the maps $h : U \rightarrow V$ and $h^{-1} : V \rightarrow U$ are of class C^k .

Take $0 \leq d \leq n$. A subset $M \subseteq \mathbb{R}^n$ is called a *manifold* of class C^k and dimension d if and only if for every $x \in M$ there are open sets $U, V \subseteq \mathbb{R}^n$, with $x \in U$ and $0 \in V$, and a diffeomorphism of class C^k $h : U \rightarrow V$ such that $h(M \cap U) = (\mathbb{R}^d \times \{0\}) \cap V$. This means that M is locally equivalent 'up to a diffeomorphism' to the d -dimensional subspace $\mathbb{R}^d \times \{0\}$ of \mathbb{R}^n . The number $n - d$ is called the *codimension* of M in \mathbb{R}^n . A practical way of proving that a subset $M \subseteq \mathbb{R}^n$ is a manifold is the following *pre-image theorem*.

Theorem 2 *Given an open set $V \subseteq \mathbb{R}^n$ and a map $f : V \rightarrow \mathbb{R}^m$ of class C^k , with $0 \leq m \leq n$, assume that $Df_x : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is surjective, for every $x \in V$ such that $f(x) = 0$. Then $f^{-1}(0) = \{x \in V : f(x) = 0\}$ is a manifold of class C^k and dimension $d = n - m$.*

When $m = 1$ the differential of f at x can be written as $Df_x(v) = \langle \nabla f(x), v \rangle$, where $\nabla f(x)$ denotes the gradient vector $\nabla f(x) = \left(\frac{\partial f}{\partial x_1}(x), \dots, \frac{\partial f}{\partial x_n}(x) \right)$. In this case, the pre-image theorem specializes as follows.

Corollary 1 *Given a map $f : V \rightarrow \mathbb{R}$ of class C^k on an open set V , define $M = f^{-1}(0)$. If $\nabla f(x) \neq 0$ for every $x \in M$, then M is a codimension one manifold of class C^k .*

This result can be easily generalized to $C^{1+\text{Lip}}$ manifolds.

Proposition 1 *Given a map $f : V \rightarrow \mathbb{R}$ of class $C^{1+\text{Lip}}$ on an open set V , define $M = f^{-1}(0)$. If $\nabla f(x) \neq 0$ for every $x \in M$, then M is a codimension one manifold of class $C^{1+\text{Lip}}$. In particular it admits an atlas consisting of charts (local diffeomorphisms) of class $C^{1+\text{Lip}}$.*

Proof Given a point $p \in M = f^{-1}(0)$, consider a surjective linear map $P : \mathbb{R}^n \rightarrow \mathbb{R}^{n-1}$ such that $\text{Ker}(P)^\perp = \text{Ker}(Df_p)$. Defining $\Phi : V \rightarrow \mathbb{R}^n$, $\Phi(x) = (f(x), P(x))$, $D\Phi_p : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is an isomorphism. Thus Φ is a mapping of class $C^{1+\text{Lip}}$ and a C^1 diffeomorphism on some small neighbourhood U of p . The inverse map $\Phi^{-1} : \Phi(U) \rightarrow U$ is of class C^1 with derivative $D\Phi_x^{-1} = (D\Phi_x)^{-1}$, and since the matrix inversion is a Lipschitz mapping, Φ^{-1} is of class $C^{1+\text{Lip}}$. Denoting by $\pi : \mathbb{R} \times \mathbb{R}^{n-1} \rightarrow \mathbb{R}^{n-1}$ the canonical projection, $\pi(t, x) = x$, the mapping $\phi = \pi \circ \Phi|_{M \cap U} : M \cap U \rightarrow \mathbb{R}^{n-1}$ is a local chart of class $C^{1+\text{Lip}}$ for M . \square

3 Main Statement

The class of r -regular sets was independently introduced in [11] and [12]. This class is also referred in [1, 2, 8–10, 14–18]. Although the details of the definitions in these papers are different, the described class is essentially the same and can be defined as follows. Denote by $B(x, r)$ the Euclidean open ball with center $x \in \mathbb{R}^n$ and radius r . Fix a positive number r and define \mathcal{U}_r as the set of all connected unions of balls $B(x, r')$ with radius $r' \geq r$. Note that, as any ball $B(x, r')$ with radius $r' \geq r$ is itself a union of balls of radius r , any set in \mathcal{U}_r is a union of balls of radius r .

Definition 1 *An open set $U \subseteq \mathbb{R}^n$ is said to be r -regular if and only if $U \in \mathcal{U}_r$ and $\overline{U}^c \in \mathcal{U}_r$.*

Let $U \subseteq \mathbb{R}^n$ be an open set.

Definition 2 *A normal vector field along ∂U is any vector field $\eta : \partial U \rightarrow \mathbb{R}^n$ such that for each $x \in \partial U$,*

$$\langle \eta(x), y - x \rangle = o(\|x - y\|) \text{ as } y \rightarrow x \text{ in } \partial U.$$

If η is a normal vector field, on a compact neighbourhood of any point $x \in \partial U$ there is a monotonic continuous function $\rho : \mathbb{R}_0^+ \rightarrow \mathbb{R}_0^+$, with $\rho(0) = 0$, such that

$$\langle \eta(x), y - x \rangle \leq \rho(\|x - y\|) \|x - y\|, \forall x, y \in \partial U.$$

The *intrinsic metric* on ∂U (see e.g. [3]), denoted by $d_{\partial U}$, is defined as follows: given $x, y \in \partial U$ and $\epsilon > 0$, let

$$d_\epsilon(x, y) = \inf_{\substack{x_0 = x, x_n = y, \\ x_i \in \partial U, \|x_{i+1} - x_i\| < \epsilon}} \sum_{i=0}^{n-1} \|x_{i+1} - x_i\| .$$

Now define

$$d_{\partial U}(x, y) = \sup_{\epsilon > 0} d_\epsilon(x, y).$$

The aim for the rest of this paper is the proof of the following characterization of r -regularity:

Theorem 3 *Let $U \subseteq \mathbb{R}^n$ be an open set. Then U is r -regular if and only if*

- (1) *there is $\eta : \partial U \rightarrow \mathbb{R}^n$ such that*
 - (i) *η is a normal vector field along ∂U ,*
 - (ii) *$\|\eta(x)\| = r$, for every $x \in \partial U$,*
 - (iii) *$\text{Lip}(\eta) \leq 1$,*
- (2) *$d_{\partial U}(x, y) \geq \frac{\pi}{2} \|x - y\|$ implies $\|x - y\| \geq 2r$, for all $x, y \in \partial U$.*

4 Local Characterization of Regularity

In this section we prove that an r -regular set admits a Lipschitz normal vector field along its boundary (see Proposition 2). We also prove that the existence of a Lipschitz normal vector field along ∂U ensures that U is locally r -regular (see Proposition 3).

Proposition 2 *Let $U \subseteq \mathbb{R}^n$ be an r -regular set. Then there is $\eta : \partial U \rightarrow \mathbb{R}^n$ such that*

- (i) *η is a normal vector field along ∂U ,*
- (ii) *$\|\eta(x)\| = r$, for every $x \in \partial U$,*
- (iii) *$\text{Lip}(\eta) \leq 1$.*

Proof Since U is r -regular, items (i), (ii) and (iii) follow from the previous paper [4]: items (i) and (ii) follow from [4, Proposition 5] and item (iii) follows from [4, Lemma 3(1)] with $v = \eta(x)$, $w = \eta(y)$ and $u = y - x$. \square

Definition 3 *An open set $U \subseteq \mathbb{R}^n$ is said to be locally r -regular if and only if for every $x \in \partial U$ there are two balls $B(x_1, r)$ and $B(x_2, r)$ tangent at x such that $B(x_1, r) \cap B(x, \epsilon) \subseteq U$ and $B(x_2, r) \cap B(x, \epsilon) \cap \bar{U} = \emptyset$, for some $\epsilon > 0$.*

Proposition 3 *Let $U \subseteq \mathbb{R}^n$ be an open set and let $\eta : \partial U \rightarrow \mathbb{R}^n$ be such that*

- (i) *η is a normal vector field along ∂U ,*
- (ii) *$\|\eta(x)\| = r$, for every $x \in \partial U$,*
- (iii) *$\text{Lip}(\eta) \leq 1$.*

Then U is locally r -regular.

The rest of this section is dedicated to the proof of Proposition 3. We assume that $\eta : \partial U \rightarrow \mathbb{R}^n$ is a normal vector field along ∂U such that $\|\eta(x)\| = r$, for every $x \in \partial U$, and $\text{Lip}(\eta) \leq 1$. First we shall prove some auxiliary lemmas.

Lemma 1 *Given $x \in \partial U$ we have that*

$$\langle \eta(x), \eta(y) - \eta(x) \rangle = o(\|x - y\|) \text{ as } y \rightarrow x \text{ in } \partial U .$$

Moreover, given $t \in (-1/2, 1/2)$, we have that

$$\|x + t\eta(x) - (y + t\eta(y))\| \geq \sqrt{1 - 2|t|} \|x - y\| ,$$

for any $x, y \in \partial U$.

Proof Since

$$\begin{aligned} \|\eta(x) - \eta(y)\|^2 &= \langle \eta(x) - \eta(y), \eta(x) - \eta(y) \rangle \\ &= 2 \langle \eta(x), \eta(x) - \eta(y) \rangle - \langle \eta(x) + \eta(y), \eta(x) - \eta(y) \rangle \\ &= 2 \langle \eta(x), \eta(x) - \eta(y) \rangle - \underbrace{(\|\eta(x)\|^2 - \|\eta(y)\|^2)}_{=0} \end{aligned}$$

we get that

$$\langle \eta(x), \eta(x) - \eta(y) \rangle = \frac{1}{2} \|\eta(x) - \eta(y)\|^2 \leq \frac{1}{2} \|x - y\|^2 ,$$

where the last inequality follows because $\text{Lip}(\eta) \leq 1$. Therefore,

$$\langle \eta(x), \eta(y) - \eta(x) \rangle = o(\|x - y\|) \text{ (} y \rightarrow x \text{)} .$$

Finally, given $t \in (-1/2, 1/2)$, we have that

$$\begin{aligned} \|x + t\eta(x) - y - t\eta(y)\|^2 &= \\ &= \|t\eta(x) - t\eta(y)\|^2 + \|x - y\|^2 \\ &\quad + 2 \langle x - y, t\eta(x) - t\eta(y) \rangle \\ &\geq \|x - y\|^2 - 2|t| \|x - y\| \|\eta(x) - \eta(y)\| \\ &\geq \|x - y\|^2 - 2|t| \|x - y\|^2 \\ &= \|x - y\|^2 (1 - 2|t|) , \end{aligned}$$

which implies that

$$\|x + t\eta(x) - y - t\eta(y)\| \geq \sqrt{1 - 2|t|} \|x - y\| ,$$

and completes the proof. \square

It follows that on a compact neighbourhood of every point $x \in \partial U$ there is a monotonic continuous function $\tilde{\rho} : \mathbb{R}_0^+ \rightarrow \mathbb{R}_0^+$, with $\rho(0) = 0$, such that

$$\langle \eta(x), \eta(y) - \eta(x) \rangle \leq \tilde{\rho}(\|x - y\|) \|x - y\|, \quad \forall x, y \in \partial U .$$

For each $\delta > 0$ we define the δ -tubular neighbourhood of ∂U , $N_\delta = \{x \in \mathbb{R}^n : d(x, \partial U) < \delta\}$.

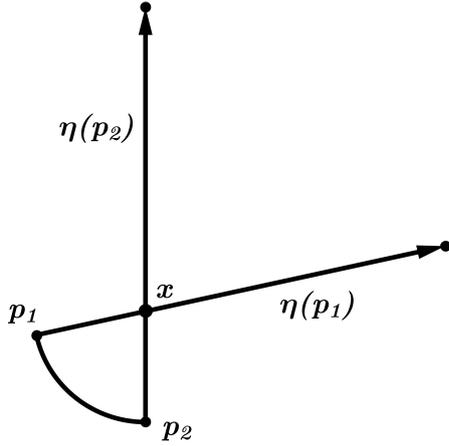


Fig. 1 Uniqueness of the projection

Lemma 2 *Given $x \in N_{r/2}(\partial U)$ there is a unique point $p \in \partial U$ such that $\|x - p\| = d(x, \partial U)$.*

Proof Assume there are two points $p_1 \neq p_2$ in ∂U such that $\|x - p_1\| = \|x - p_2\| = d(x, \partial U)$, where $x \in N_{r/2}(\partial U)$. Then since $\|x - p_1\| = \|x - p_2\| < r/2$, we have (see Fig. 1),

$$\begin{aligned} \|\eta(p_1) - \eta(p_2)\| &= \frac{r}{d(x, \partial U)} \|(x - p_1) - (x - p_2)\| \\ &= \frac{r}{d(x, \partial U)} \|p_1 - p_2\| \\ &\geq \frac{r}{r/2} \|p_1 - p_2\| = 2 \|p_1 - p_2\| , \end{aligned}$$

which contradicts $\text{Lip}(\eta) \leq 1$, unless $p_1 = p_2$. \square

By the previous lemma we can define a projection $\pi : N_{r/2}(\partial U) \rightarrow \partial U$ such that $\|x - \pi(x)\| = d(x, \partial U)$, for every $x \in N_{r/2}(\partial U)$.

Lemma 3 *Given any $0 < s < \frac{1}{2}$, we have that the mapping $\pi : N_{sr}(\partial U) \rightarrow \partial U$ is a Lipschitz projection with $\text{Lip}(\pi) \leq \frac{1}{\sqrt{1-2s}}$, and for every $x \in \partial U$,*

$$\pi^{-1}(x) = \{x + t\eta(x) : t \in (-s, s)\} .$$

Proof The second part is clear, hence we only need to prove that π is Lipschitz. Given $x, y \in N_{sr}(\partial U)$, we have $x = \pi(x) + t_1\eta(\pi(x))$ and $y = \pi(y) + t_2\eta(\pi(y))$, with $t_1, t_2 \in (-s, s)$. Let x', y' be such that $x' = \pi(x) + t\eta(\pi(x))$ and $y' = \pi(y) + t\eta(\pi(y))$, for some $t \in (-s, s)$, and such that $\|x' - y'\| \leq \|x - y\|$. Clearly, $\pi(x) = \pi(x')$ and $\pi(y) = \pi(y')$. Using the almost orthogonality relation of Lemma 1, we obtain that

$$\begin{aligned} \|\pi(x) - \pi(y)\| &= \|\pi(x') - \pi(y')\| \\ &\leq \frac{1}{\sqrt{1-2|t|}} \|(\pi(x) + t\eta(\pi(x))) - (\pi(y) + t\eta(\pi(y)))\| \\ &= \frac{1}{\sqrt{1-2|t|}} \|x' - y'\| \leq \frac{1}{\sqrt{1-2|t|}} \|x - y\| \\ &\leq \frac{1}{\sqrt{1-2s}} \|x - y\| . \end{aligned}$$

\square

Define now the function $f : N_{r/2}(\partial U) \rightarrow \mathbb{R}$,

$$f(x) = \langle x - \pi(x), \eta(\pi(x)) \rangle .$$

Proposition 4 *The function f is of class C^1 with differential given by*

$$Df_x(v) = \langle v, \eta(\pi(x)) \rangle .$$

Proof Next argument is a straightforward adaptation of the proof of Proposition 8 in [4], which we include here for the sake of completeness. We must show that one has for every $x \in N_{r/2}(\partial U)$

$$f(y) - f(x) - \langle y - x, \eta(\pi(x)) \rangle = o(\|x - y\|) \quad (y \rightarrow x)$$

i.e., given $x \in N_{r/2}(\partial U)$ and $\epsilon > 0$ there is $\delta > 0$ such that for every $y \in N_{r/2}(\partial U)$ with $\|x - y\| \leq \delta$ one has

$$|f(y) - f(x) - \langle y - x, \eta(\pi(x)) \rangle| \leq \epsilon \|x - y\| .$$

To simplify the notation we shall omit some parentheses, writing for instance πx instead of $\pi(x)$.

$$\begin{aligned}
& |f(y) - f(x) - \langle y - x, \eta(\pi x) \rangle| = \\
& = |\langle y - \pi y, \eta(\pi y) \rangle - \langle x - \pi x, \eta(\pi x) \rangle - \langle y - x, \eta(\pi x) \rangle| \\
& = |\langle y - \pi y, \eta(\pi y) \rangle - \langle y - \pi x, \eta(\pi x) \rangle| \\
& = |\langle y - \pi y, \eta(\pi y) \rangle - \langle y - \pi x, \eta(\pi y) \rangle| \\
& \quad + |\langle y - \pi x, \eta(\pi y) \rangle - \langle y - \pi x, \eta(\pi x) \rangle| \\
& = |\langle \pi x - \pi y, \eta(\pi y) \rangle| + |\langle y - \pi x, \eta(\pi y) - \eta(\pi x) \rangle| \\
& = |\langle \pi x - \pi y, \eta(\pi y) \rangle| + |\langle y - x, \eta(\pi y) - \eta(\pi x) \rangle| \\
& \quad + |\langle x - \pi x, \eta(\pi y) - \eta(\pi x) \rangle| \\
& = |\langle \pi x - \pi y, \eta(\pi y) \rangle| + |\langle y - x, \eta(\pi y) - \eta(\pi x) \rangle| \\
& \quad + \frac{\|x - \pi x\|}{r} |\langle \eta(\pi x), \eta(\pi y) - \eta(\pi x) \rangle| \\
& \leq \rho(\|\pi x - \pi y\|) \|\pi x - \pi y\| + \|x - y\| \|\eta(\pi x) - \eta(\pi y)\| \\
& \quad + \tilde{\rho}(\|\pi x - \pi y\|) \|\pi x - \pi y\| \\
& \leq C\rho(\|\pi x - \pi y\|) \|x - y\| + C \|x - y\|^2 + C\tilde{\rho}(\|x - y\|) \\
& \leq C \|x - y\| (\rho(\|\pi x - \pi y\|) + \|x - y\| + \tilde{\rho}(\|\pi x - \pi y\|)) \\
& \leq C \|x - y\| \hat{\rho}(\|x - y\|),
\end{aligned}$$

where $\hat{\rho}(t) = t + \rho(t) + \tilde{\rho}(t)$. We observe that in the penultimate step we have used the fact that both η and π are Lipschitz on N_{sr} , provided that $s \in (0, 1/2)$, and the constant $C = C_s$ is given explicitly by $C_s = \frac{1}{\sqrt{1-2s}}$. \square

Remark 1 In Proposition 4, function f is indeed of class $C^{1+\text{Lip}}$. Just note that, since η and π are Lipschitz, the differential Df is also Lipschitz.

We are now ready to prove Proposition 3, i.e., to show that the level set $\partial U = f^{-1}(0)$ is locally bounded between two spheres of radius r tangent at x , for any $x \in \partial U$.

Proof (of Proposition 3) By the mean value theorem, for any $x, x + v \in N_{r/2}(\partial U)$ we have that

$$f(x + v) - f(x) - Df_x(v) = \int_0^1 Df_{x+tv}(v) dt - Df_x(v).$$

But since, for any $x, y \in N_{r/2}(\partial U)$, we have that

$$\begin{aligned}
|Df_x(v) - Df_y(v)| & = |\langle v, \eta(\pi x) \rangle - \langle v, \eta(\pi y) \rangle| \\
& = |\langle v, \eta(\pi x) - \eta(\pi y) \rangle| \\
& \leq \|v\| \|\pi x - \pi y\| \\
& \leq \text{Lip}(\pi) \|v\| \|x - y\|,
\end{aligned}$$

it follows that

$$\begin{aligned}
|f(x + v) - f(x) - Df_x(v)| & = \\
& = \left| \int_0^1 (Df_{x+tv}(v) - Df_x(v)) dt \right| \\
& \leq \int_0^1 |Df_{x+tv}(v) - Df_x(v)| dt \\
& \leq \int_0^1 \text{Lip}(\pi) \|v\| \|x + tv - x\| dt \\
& = \text{Lip}(\pi) \|v\|^2 \int_0^1 t dt \\
& = \frac{1}{2} \text{Lip}(\pi) \|v\|^2.
\end{aligned}$$

Therefore, given $x \in \partial U$ and $x + v \in N_{sr}$, for some $s \in (0, 1/2)$, we get that

$$\begin{aligned}
|f(x + v) - f(x) - Df_x(v)| & = \\
& = |f(x + v) - Df_x(v)| \leq \frac{C_s}{2} \|v\|^2,
\end{aligned}$$

or, equivalently,

$$|f(x + v) - \langle v, \eta(x) \rangle| \leq \frac{C_s}{2} \|v\|^2, \quad (1)$$

where $C_s = \frac{1}{\sqrt{1-2s}}$.

Now define the set $A(x)$ of vectors $v \in \mathbb{R}^n$ such that

$$\left\| (x + v) - \left(x \pm \frac{\eta(x)}{C_s} \right) \right\|^2 < \left(\frac{r}{C_s} \right)^2,$$

for one of the signs $+$ or $-$. Note that $x + v$ belongs to one of the two balls $B\left(x \pm \frac{\eta(x)}{C_s}, \frac{r}{C_s}\right)$, which are tangent at x , if and only if $v \in A(x)$. We claim that $v \in A(x)$ implies that

$$|\langle v, \eta(x) \rangle| > \frac{C_s}{2} \|v\|^2.$$

In fact, first note that

$$\begin{aligned}
\left\| (x + v) - \left(x \pm \frac{\eta(x)}{C_s} \right) \right\|^2 & \geq \\
\|v\|^2 + \left(\frac{\|\eta(x)\|}{C_s} \right)^2 & - \frac{2}{C_s} |\langle v, \eta(x) \rangle|.
\end{aligned}$$

Consequently, if $v \in A(x)$, then the following inequality holds

$$\left(\frac{r}{C_s} \right)^2 > \|v\|^2 + \left(\frac{\|\eta(x)\|}{C_s} \right)^2 - \frac{2}{C_s} |\langle v, \eta(x) \rangle|$$

which is equivalent to

$$|\langle v, \eta(x) \rangle| > \frac{C_s}{2} \|v\|^2.$$

Therefore, if $v \in A(x)$, then inequality (1) implies that

$$|f(x+v)| \geq |\langle v, \eta(x) \rangle| - \frac{C_s}{2} \|v\|^2 > 0.$$

Hence,

$$\{x+v : v \in A(x)\} \cap f^{-1}(0) = \emptyset$$

and thus, we may conclude that locally the level set $\partial U = f^{-1}(0)$ is bounded between two balls of radius r/C_s tangent at x . Notice that the constant C_s gets close to 1 as $s \rightarrow 0$. Therefore, decreasing the neighbourhood N_{sr} of ∂U , we can take the radius of the balls arbitrarily close to r . \square

5 An Example

Let $U \subseteq \mathbb{R}^n$ be an open set. In this section we show that the existence of a normal vector field of constant norm r and Lipschitz constant 1 along ∂U (given by items (1) (i), (ii) and (iii) of Theorem 3) is not enough to guarantee that U is (globally) r -regular. Indeed, consider a body U , as sketched in Fig. 2, but in a large scale so that the curvature is small, but the connection between the balloons is still very thin. Then the Lipschitz assumption on the “normal vector field” is satisfied, but U is not r -regular because of the narrow connection.

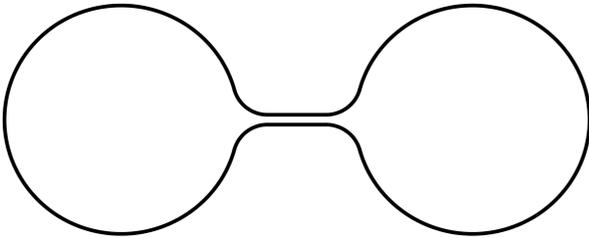


Fig. 2 A non-regular set that satisfies the Lipschitz assumption on the normal vector field

Therefore, we need an extra assumption to ensure that U is (globally) r -regular. This extra assumption (given by item (2) of Theorem 3) is that for every $x, y \in \partial U$, $d_{\partial U}(x, y) \geq \frac{\pi}{2} d(x, y)$ implies $d(x, y) \geq 2r$. The bound $\frac{\pi}{2}$ corresponds to the maximum ratio between these two distances when U is an Euclidean ball.

6 $C^{1+\text{Lip}}$ -geodesics

In the proof of the “if” part of Theorem 3 we shall use some tools and results from Riemannian Geometry. The natural scope of this theory is that of smooth manifolds

of at least class C^2 . The results we need can easily be generalized to $C^{1+\text{Lip}}$ manifolds, but since we are not aware of any suitable reference, for the sake of completeness, we include this section where we prove the needed generalizations.

From now on $M \subseteq \mathbb{R}^n$ will denote a class $C^{1+\text{Lip}}$ connected manifold of dimension $n - 1$, defined as a regular level set of some function of class $C^{1+\text{Lip}}$. By Proposition 1, given $p, q \in M$ there are $C^{1+\text{Lip}}$ curves $\gamma : [a, b] \rightarrow M$ such that $\gamma(a) = p$ and $\gamma(b) = q$. Hence the *loop space* $\Omega = \Omega_{p,q}(a, b)$ of all such curves is non empty. This loop space is a subset of the normed vector space $C^{1+\text{Lip}}([a, b], \mathbb{R}^n)$ endowed with the norm

$$\|\gamma\|_{C^1} = \max_{t \in [a, b]} \|\gamma(t)\| + \max_{t \in [a, b]} \|\gamma'(t)\|.$$

We topologize Ω as a subspace of this normed space. The *energy* and *length* of a curve $\gamma \in \Omega$ are respectively defined to be

$$E(\gamma) = \int_a^b \|\gamma'(t)\|^2 dt \quad \text{and} \quad L(\gamma) = \int_a^b \|\gamma'(t)\| dt.$$

Fix a curve $\gamma_0 \in \Omega$. We call *variation of γ_0* to any continuous function $h : (-\delta, \delta) \times [a, b] \rightarrow M$ such that

- (1) $h(s, t)$ is Lipschitz on $(-\delta, \delta) \times [a, b]$,
- (2) $h(0, t) = \gamma_0(t)$ for all $t \in [a, b]$,
- (3) $t \mapsto h(s, t)$ is of class $C^{1+\text{Lip}}$ for every $s \in (-\delta, \delta)$,
- (4) the function $\zeta(t) := \frac{dh}{ds}(0, t)$ is Lipschitz in $[a, b]$.

We shall also say that h is a *variation of γ_0 along ζ* . The variation h is said to be *proper* if furthermore

- (5) $h(s, a) = p$ and $h(s, b) = q$ for all $s \in (-\delta, \delta)$.

The variation h determines a curve $\bar{h} : (-\delta, \delta) \rightarrow \Omega$ such that $\bar{h}(0) = \gamma_0$. The vector field ζ along γ_0 is the formal derivative $\bar{h}'(0) = \zeta$ of this curve. It is a section of the vector bundle $\gamma_0^* TM$ over $[a, b]$, which has fiber $T_{\gamma_0(t)} M$ at the base point $t \in [a, b]$. We shall also refer to ζ as a *tangent vector field*, to emphasize that it is tangent to the hypersurface M . If h is a proper variation then the vector field ζ is also proper in the sense that $\zeta(a) = \zeta(b) = 0$.

The existence of variations of a curve $\gamma_0 \in \Omega$ is usually proven through the *exponential map* of the Riemannian manifold M . We avoid these technicalities working in a local chart. Given $\varepsilon > 0$, two vector fields $\zeta_1, \zeta_2 : [a, b] \rightarrow \mathbb{R}^n$ are said to be ε -close when $\|\zeta_1(t) - \zeta_2(t)\| \leq \varepsilon$ for all $t \in [a, b]$.

Proposition 5 *Given $\varepsilon > 0$, a curve $\gamma_0 \in \Omega$ contained in the domain of a single chart of M , and any continuous proper vector field $\zeta : [a, b] \rightarrow \mathbb{R}^n$ tangent to M along γ_0 , there is at least one proper variation of γ_0 along a Lipschitz vector field ε -close to ζ .*

Proof Fix a curve $\gamma_0 \in \Omega$ for which there is a local chart $\phi : U \subseteq M \rightarrow \mathbb{R}^{n-1}$ of class $C^{1+\text{Lip}}$ such that $\gamma_0([a, b]) \subset U$. Define

$$\tilde{\zeta}(t) := D\phi_{\gamma_0(t)}\zeta(t),$$

and

$$\tilde{\gamma}_0(t) := \phi(\gamma_0(t)).$$

Let $\tilde{\zeta}^*$ be a proper C^∞ -vector field ε -close to $\tilde{\zeta}$. We define the function $\tilde{h} : (-\delta, \delta) \times [a, b] \rightarrow \mathbb{R}^{n-1}$ by

$$\tilde{h}(s, t) := \tilde{\gamma}_0(t) + s\tilde{\zeta}^*(t).$$

Finally, let

$$h(s, t) := \phi^{-1}\tilde{h}(s, t) = \phi^{-1}(\tilde{\gamma}_0(t) + s\tilde{\zeta}^*(t)).$$

We have that

- (1) $h(s, t)$ is Lipschitz on $(-\delta, \delta) \times [a, b]$, because $\tilde{\gamma}_0$ and ϕ^{-1} are of class $C^{1+\text{Lip}}$ and $\tilde{\zeta}^*$ is of class C^∞ ,
- (2) $h(0, t) = \phi^{-1}(\tilde{\gamma}_0(t)) = \gamma_0(t)$ for all $t \in [a, b]$,
- (3) $t \mapsto h(s, t)$ is of class $C^{1+\text{Lip}}$ for every $s \in (-\delta, \delta)$, because $\tilde{\gamma}_0$ and ϕ^{-1} are of class $C^{1+\text{Lip}}$ and $\tilde{\zeta}^*$ is of class C^∞ ,
- (4) the function $\zeta^*(t) := \frac{dh}{ds}(0, t) = D\phi_{\tilde{\gamma}_0(t)}^{-1}\tilde{\zeta}^*(t)$ is Lipschitz in $[a, b]$,
- (5) $h(s, a) = \phi^{-1}(\tilde{\gamma}_0(a)) = p$ and $h(s, b) = \phi^{-1}(\tilde{\gamma}_0(b)) = q$ for all $s \in (-\delta, \delta)$, because $\tilde{\zeta}^*$ is proper.

Therefore, h is a proper variation of γ_0 along ζ^* . Furthermore,

$$\begin{aligned} \|\zeta^*(t) - \zeta(t)\| &= \left\| D\phi_{\tilde{\gamma}_0(t)}^{-1}\tilde{\zeta}^*(t) - \zeta(t) \right\| \\ &\leq K \left\| \tilde{\zeta}^*(t) - \tilde{\zeta}(t) \right\|, \\ &\leq K\varepsilon \end{aligned}$$

where $K = \max_{x \in \phi(U)} \|D\phi_x^{-1}\|$. We have $K < +\infty$ because we can shrink the domain U to a compact neighbourhood of $\gamma_0([a, b])$. Since $K\varepsilon$ can be arbitrarily small the lemma follows. \square

Proposition 6 *The length functional $L : \Omega \rightarrow \mathbb{R}$ attains an absolute minimum at a curve $\gamma \in \Omega$ which is also an absolute minimum of the energy functional $E : \Omega \rightarrow \mathbb{R}$. Moreover this minimum satisfies*

- (1) $\|\gamma'(t)\|$ is constant,
- (2) $\gamma''(t) \perp T_{\gamma(t)}M$ for almost every $t \in [a, b]$.

Proof For any $c > 0$ define Ω^c to be the set of curves $\gamma \in \Omega = \Omega_{p,q}(a, b)$ such that $\text{Lip}(\gamma') \leq c$. Fix $c > 0$ such that $\Omega^c \neq \emptyset$. It is readily seen that the set of derivatives $\{\gamma' : \gamma \in \Omega^c\}$ is equicontinuous. Hence by the Ascoli-Arzelá's theorem the set Ω^c is compact.

Since the functional L is continuous on Ω , by Weierstrass Minimum Principle there is a curve $\gamma_0 \in \Omega^c$ such that $L(\gamma_0)$ is the minimum value of the length on Ω^c . The trace $\Gamma = \gamma_0([a, b])$ is a 1-dimensional manifold of class C^1 , for otherwise it wouldn't minimize the length. Hence we can reparametrize γ_0 to have constant speed. Thus we assume γ_0 satisfies $\|\gamma_0'(t)\| = c_0$ for all $t \in [a, b]$, with $c_0 \leq c$. Of course γ_0 is also the length absolute minimum in Ω . By Jensen's inequality, for any other curve $\gamma \in \Omega$ we have

$$\begin{aligned} E(\gamma_0) &= c^2(b-a) = \frac{1}{b-a} L(\gamma_0)^2 \leq \frac{1}{b-a} L(\gamma)^2 \\ &= (b-a) \left(\frac{1}{b-a} \int_a^b \|\gamma'(t)\| dt \right)^2 \\ &\leq \int_a^b \|\gamma'(t)\|^2 dt = E(\gamma), \end{aligned}$$

which proves that γ_0 is also an absolute minimum of $E : \Omega \rightarrow \mathbb{R}$.

Let us now prove item (2). We can split γ in several pieces so that each part is contained in the domain of a single chart. Clearly each subcurve of γ also minimizes both L and E . Thus, since conclusion (2) of Proposition 6 is local, we can assume that $\gamma([a, b])$ is contained in the domain of a single chart. This will allow us to apply Proposition 5. We claim that for any continuous proper vector field ξ tangent to M along γ

$$\int_a^b \langle \gamma''(t), \xi(t) \rangle dt = 0.$$

We first prove this for Lipschitz vector fields associated to proper variations of γ . Let h be a proper variation of γ along the Lipschitz vector field ξ . Because $E(\bar{h}(s))$ attains its minimum value at $s = 0$,

$$\begin{aligned} 0 &= \frac{d}{ds} E(\bar{h}(s))|_{s=0} = \int_a^b \frac{d}{ds} \left\langle \frac{dh}{dt}, \frac{dh}{dt} \right\rangle dt \\ &= 2 \int_a^b \left\langle \gamma'(t), \frac{d}{dt} \left[\frac{dh}{ds}(0, t) \right] \right\rangle dt \\ &= 2 \int_a^b \langle \gamma'(t), \xi'(t) \rangle dt \\ &= [\langle \gamma'(t), \xi(t) \rangle]_a^b - \int_a^b \langle \gamma''(t), \xi(t) \rangle dt. \end{aligned}$$

Thus, since ξ is proper, $[\langle \gamma'(t), \xi(t) \rangle]_a^b = 0$ and

$$\int_a^b \langle \gamma''(t), \xi(t) \rangle dt = 0.$$

Consider now any continuous proper vector field $\zeta(t)$ tangent to M along γ , and fix a small number $\varepsilon > 0$. By Proposition 5, there is a proper variation h of γ along

a Lipschitz vector field ξ which is ε -close to ζ . From the previous claim we know that $\int_a^b \langle \gamma''(t), \xi(t) \rangle dt = 0$. Therefore

$$\begin{aligned} \left| \int_a^b \langle \gamma''(t), \zeta(t) \rangle dt \right| &\leq \left| \int_a^b \langle \gamma''(t), \zeta(t) \rangle - \langle \gamma''(t), \xi(t) \rangle dt \right| \\ &\leq \int_a^b |\langle \gamma''(t), \zeta(t) - \xi(t) \rangle| dt \\ &\leq \int_a^b \|\gamma''(t)\| \|\zeta(t) - \xi(t)\| dt \\ &\leq c(b-a)\varepsilon, \end{aligned}$$

where $c = \text{Lip}(\gamma')$. Consequently, since ε can be arbitrarily small,

$$\int_a^b \langle \gamma''(t), \zeta(t) \rangle dt = 0.$$

To finish the proof we need to establish (2). Consider continuous proper vector fields $\zeta_1, \dots, \zeta_{n-1}$ along γ , tangent to M , such that $\{\zeta_1(t), \dots, \zeta_{n-1}(t)\}$ is a basis of $T_{\gamma(t)}M$ for all $t \in (a, b)$. Given any continuous function $g : [a, b] \rightarrow \mathbb{R}$, since $g\zeta_j$ is a continuous proper vector field we have $\int_a^b \langle \gamma''(t), g(t)\zeta_j(t) \rangle dt = 0$, which implies that

$$\int_a^b \langle \gamma''(t), \zeta_j(t) \rangle g(t) dt = 0.$$

Thus, because g is arbitrary,

$$\langle \gamma''(t), \zeta_j(t) \rangle = 0 \quad \forall 1 \leq j \leq n-1,$$

for almost every $t \in [a, b]$, which implies that $\gamma''(t) \perp T_{\gamma(t)}M$ for the same values of $t \in [a, b]$. \square

By the previous proposition given points $p, q \in M$, there is a curve $\gamma \in \Omega$ connecting p and q , of minimum length $\ell = L(\gamma)$, which can be reparametrized on the interval $[0, \ell]$ to have unit speed, i.e., $\|\gamma'(t)\| = 1$ for every $t \in [0, \ell]$. As usual, such a curve will be referred as *minimizing unit geodesic* from p to q .

Proposition 7 *If there is a Lipschitz normal field η to M with constant norm, $\|\eta(x)\| = r$ for every $x \in M$, and Lipschitz constant $\text{Lip}(\eta) \leq 1$, then every minimizing unit geodesic $\gamma : [0, \ell] \rightarrow M$ satisfies $\text{Lip}(\gamma') \leq 1/r$.*

Proof Since η is a normal field we have $\langle \gamma'(t), \eta(\gamma(t)) \rangle = 0$ for every t . Both factors in this product are Lipschitz. By Theorem 1 (Rademacher's theorem) these functions are differentiable almost everywhere. Applying Leibnitz rule at points where both these functions are differentiable we get $\langle \gamma''(t), \eta(\gamma(t)) \rangle + \langle \gamma'(t), (\eta \circ \gamma)'(t) \rangle = 0$.

On the other hand, by Proposition 6 we have that $\gamma''(t)$ is collinear with $\eta(\gamma(t))$ for almost every t . But since this orthogonality is equivalent to the identity $\gamma''(t) = \frac{1}{r^2} \langle \gamma''(t), \eta(\gamma(t)) \rangle \eta(\gamma(t))$, it follows that

$$\gamma''(t) = -\frac{1}{r^2} \langle \gamma'(t), (\eta \circ \gamma)'(t) \rangle \eta(\gamma(t))$$

for almost every t . Now, by assumption $\|\gamma'(t)\| = 1$ and $\|\eta(\gamma(t))\| = r$ for all t . Since $\text{Lip}(\eta) \leq 1$, we must have $\text{Lip}(\eta \circ \gamma) \leq 1$, and hence $\|(\eta \circ \gamma)'(t)\| \leq 1$ for almost every t . Therefore

$$\|\gamma''(t)\| \leq \frac{1}{r^2} \|\gamma'(t)\| \|(\eta \circ \gamma)'(t)\| \|\eta(\gamma(t))\| \leq \frac{1}{r}$$

for almost every t . Finally, by the mean value theorem

$$\begin{aligned} \|\gamma'(t) - \gamma'(t_0)\| &= \left\| \int_{t_0}^t \gamma''(s) ds \right\| \leq \int_{t_0}^t \|\gamma''(s)\| ds \\ &\leq \int_{t_0}^t \frac{1}{r} ds \leq \frac{1}{r} |t - t_0|, \end{aligned}$$

which proves that $\text{Lip}(\gamma') \leq 1/r$. \square

7 A Sturm-Liouville Lemma

In this section we prove a Sturm-Liouville lemma that we will use, in the next section, in the proof of the “if” part of Theorem 3.

Lemma 4 *Consider a curve $\gamma : [0, +\infty[\rightarrow \mathbb{R}^n$ of class $C^{1+\text{Lip}}$ such that*

- (a) $\|\gamma(0)\| = r$,
- (b) $\langle \gamma(0), \gamma'(0) \rangle = 0$,
- (c) $\|\gamma'(t)\| = 1$, for all $t \geq 0$,
- (d) $\text{Lip}(\gamma') \leq 1/r$.

Then for every $t \in [0, \pi r]$, $\|\gamma(t)\| \geq r$.

Proof First define, for every $t \in [0, +\infty[$,

$$\varphi(t) := \langle \gamma(t), \gamma(t) \rangle = \|\gamma(t)\|^2.$$

We have that

$$\varphi'(t) = 2\langle \gamma(t), \gamma'(t) \rangle.$$

By Theorem 1 (Rademacher's theorem) function γ' is differentiable almost everywhere. Furthermore, we have that $\|\gamma''(t)\| \leq 1/r$ for almost every t , because $\text{Lip}(\gamma') \leq 1/r$. Therefore, for almost every t , we can obtain that

$$\begin{aligned} \varphi''(t) &= 2(\|\gamma'(t)\|^2 + \langle \gamma(t), \gamma''(t) \rangle) \\ &\geq 2(1 - \|\gamma(t)\| \|\gamma''(t)\|) \\ &\geq 2\left(1 - \frac{\|\gamma(t)\|}{r}\right) \\ &= 2\left(1 - \sqrt{\frac{\varphi(t)}{r^2}}\right) \\ &\geq 2\left(1 - \left(1 + \frac{1}{2}\left(\frac{\varphi(t)}{r^2} - 1\right)\right)\right) \\ &= 1 - \frac{\varphi(t)}{r^2}. \end{aligned}$$

Now define

$$\psi(t) := 1 - \frac{\varphi(t)}{r^2}.$$

Derivating ψ and using the above inequality we get that, for almost every t ,

$$\psi''(t) = -\frac{1}{r^2}\varphi''(t) \leq -\frac{1}{r^2}\psi(t)$$

which is equivalent to

$$\psi''(t) + \frac{1}{r^2}\psi(t) \leq 0.$$

Moreover, it is clear that,

$$\psi(t) < 0 \Leftrightarrow \varphi(t) > r^2,$$

or equivalently,

$$\psi(t) < 0 \Leftrightarrow \|\gamma(t)\| > r.$$

Let $b > 0$ be the first time $t > 0$ such that $\|\gamma(t)\| = r$, with $b = +\infty$ if no such time exists. Our goal is to show that $b \geq \pi r$, something obvious if $b = +\infty$. Hence we assume that b is finite. By definition we have $\psi(t) < 0$ for all $t \in (0, b)$. Define on this interval

$$q(t) := \frac{\psi''(t)}{\psi(t)} + \frac{1}{r^2}.$$

Assumptions (c) and (d) imply that φ'' , and hence ψ'' , are measurable bounded functions. Therefore $q(t)$ is locally integrable on $(0, b)$. Clearly $q(t) \geq 0$ for all $t \in (0, b)$ and

$$\psi''(t) + \left(\frac{1}{r^2} - q(t)\right)\psi(t) = 0.$$

Since $\varphi(0) = \varphi(b) = r^2$ there is a first time $0 < t_0 < b$ such that $\langle \gamma(t_0), \gamma'(t_0) \rangle = \frac{1}{2}\varphi'(t_0) = 0$. Clearly, $\psi(t_0) < 0$ and $\psi'(t_0) = 0$. By Sturm-Liouville theory, see [7], we can compare the solution of the previous equation with the solution of the problem

$$\begin{cases} \tilde{\psi}''(t) + \frac{1}{r^2}\tilde{\psi}(t) = 0, \\ \tilde{\psi}(t_0) = \psi(t_0) \\ \tilde{\psi}'(t_0) = \psi'(t_0) \end{cases}$$

which is given by $\tilde{\psi}(t) = \psi(t_0) \cos\left(\frac{t-t_0}{r}\right)$, to conclude that

$$\psi(t) \leq \tilde{\psi}(t),$$

while $\tilde{\psi}(t) < 0$. Therefore, the first zero b of ψ satisfies

$$b \geq t_0 + r\frac{\pi}{2}.$$

Now let,

$$\psi_1(t) := \psi(-t).$$

Clearly, $\psi_1'(t) = -\psi'(-t)$ and $\psi_1''(t) = \psi''(-t)$. Therefore, we can replace in the above differential equations, ψ by ψ_1 . Consequently, we can repeat the same arguments as before for $t \leq t_0$ to obtain that

$$0 \leq t_0 - r\frac{\pi}{2}.$$

Hence $b \geq \pi r$, which completes the proof. \square

8 Deriving Regularity

In this section we shall prove the ‘if’ part of Theorem 3, i.e., we prove the following proposition.

Proposition 8 *Given an open set $U \subseteq \mathbb{R}^n$ assume that*

- (1) *there is $\eta : \partial U \rightarrow \mathbb{R}^n$ such that*
 - (i) *η is a normal vector field along ∂U ,*
 - (ii) *$\|\eta(x)\| = r$, for every $x \in \partial U$,*
 - (iii) *$\text{Lip}(\eta) \leq 1$,*
- (2) *$d_{\partial U}(x, y) \geq \frac{\pi}{2} \|x - y\|$ implies $\|x - y\| \geq 2r$, for all $x, y \in \partial U$.*

Then U is r -regular.

Proof By Proposition 3, U is locally r -regular. Therefore, we are left to prove that the level set $\partial U = f^{-1}(0)$ is globally bounded between two balls of radius r tangent at x , for any $x \in \partial U$. Let $\eta : \partial U \rightarrow \mathbb{R}^n$ be such that conditions (1) (i), (ii) and (iii) are satisfied. Suppose that η points outward U . We want to show that $B(x + \eta(x), r) \subseteq \bar{U}^c$ and $B(x - \eta(x), r) \subseteq U$. We shall prove the first inclusion, since the proof of the second one is analogous. Suppose, by contradiction, that there exists $y \in \partial U$ such that $y \in B(x + \eta(x), r)$. Let $\gamma : [0, \ell] \rightarrow \mathbb{R}^n$ be a minimizing unit geodesic from x to y . By Proposition 7, $\text{Lip}(\gamma') \leq 1/r$. Therefore, it follows from Lemma 4 that

$$d_{\partial U}(x, y) = \ell \geq \pi r \geq \frac{\pi}{2} \|x - y\|.$$

Consequently, condition (2) implies that $\|x - y\| \geq 2r$, which is a contradiction. \square

9 Global Characterization of Regularity

To finish the proof of Theorem 3 we are left to prove the following proposition.

Proposition 9 *Let $U \subseteq \mathbb{R}^n$ be an r -regular set. Then we have that*

$$d_{\partial U}(x, y) \geq \frac{\pi}{2} \|x - y\| \text{ implies } \|x - y\| \geq 2r,$$

for all $x, y \in \partial U$.

Lemma 5 Let $U \subseteq \mathbb{R}^n$ be an r -regular set. The projection $\pi : N_r \rightarrow \partial U$ that to each point $x \in N_r$ associates the nearest point $\pi(x) \in \partial U$ is continuous.

Proof The minimizing projection π is well defined by [4, Lemma 5]. Consider any sequence $x_n \in N_r$ such that $x_n \rightarrow x$ with $x \in N_r$ and suppose, by contradiction, that $\pi(x_n) \not\rightarrow \pi(x)$. Therefore, there exists some subsequence x_{n_k} such that $\pi(x_{n_k}) \rightarrow y$ with $y \in \partial U$ and $y \neq \pi(x)$. But then

$$\|x - \pi(x)\| = \lim_n \|x_{n_k} - \pi(x_{n_k})\| = \|x - y\|,$$

which implies that $\pi(x) = y$. \square

Lemma 6 Let $U \subseteq \mathbb{R}^n$ be an r -regular set. Given $x, y \in \partial U$ such that $\|x - y\| < 2r$, if B denotes the closed ball with diameter $[x, y]$ then the subspace $\partial U \cap B$ is connected.

Proof Take $x, y \in \partial U$ such that $\|x - y\| < 2r$. We claim that $[x, y] \subseteq N_r$. Given $z \in [x, y]$, assume $\|x - z\| \leq \|y - z\|$. Then

$$d(z, \partial U) \leq \|x - z\| \leq \frac{1}{2} \|x - y\| < r,$$

which proves that $z \in N_r$. Otherwise, if $\|x - z\| \geq \|y - z\|$ then

$$d(z, \partial U) \leq \|y - z\| \leq \frac{1}{2} \|x - y\| < r,$$

which again proves that $z \in N_r$. Hence $[x, y] \subseteq N_r$, and $\pi([x, y])$ is a connected curve joining x and y in ∂U . To finish we just need to show that $\pi([x, y]) \subseteq B$. Let $p = \frac{x+y}{2}$ be the centre of B and notice that B has radius $r = \|x - p\| = \|y - p\|$. Given $z \in [x, y]$, assume $\|x - z\| \leq \|y - z\|$. Then

$$\begin{aligned} \|\pi(z) - p\| &\leq \|\pi(z) - z\| + \|z - p\| \\ &\leq \|x - z\| + \|z - p\| \\ &= \|x - p\|, \end{aligned}$$

which proves that $\pi(z) \in B$. Otherwise, if $\|x - z\| \geq \|y - z\|$ then

$$\begin{aligned} \|\pi(z) - p\| &\leq \|\pi(z) - z\| + \|z - p\| \\ &\leq \|y - z\| + \|z - p\| \\ &= \|y - p\|, \end{aligned}$$

which again proves that $\pi(z) \in B$. \square

Corollary 2 Let $U \subseteq \mathbb{R}^n$ be an r -regular set. Given $x, y \in \partial U$ such that $0 < \|x - y\| < 2r$ there is $z \in \partial U$ satisfying

(a) $\langle z - \frac{x+y}{2}, y - x \rangle = 0$,

(b) $\|z - \frac{x+y}{2}\| \leq \frac{1}{2} \|x - y\|$,

(c) the line through z normal to ∂U is contained in the plane determined by the points x, y and z .

Proof Let $\delta = \frac{1}{2} \|x - y\|$ and consider the continuous function $f : \partial U \cap \overline{B}_\delta(\frac{x+y}{2}) \rightarrow \mathbb{R}$ defined by $f(z) = \langle z - \frac{x+y}{2}, y - x \rangle$. Remark that $f(x) < 0$ and $f(y) > 0$. Thus, since $\partial U \cap \overline{B}_\delta(\frac{x+y}{2})$ is connected, $f(z) = 0$ for some $z \in \partial U \cap \overline{B}_\delta(\frac{x+y}{2})$. Let H denote the hyperplane $\{z \in \mathbb{R}^n : f(z) = 0\}$. We know there is at least one point $z \in \partial U \cap H$ satisfying $\|z - \frac{x+y}{2}\| \leq \delta$. Hence we can pick such a point z minimizing the distance $\|z - \frac{x+y}{2}\|$. It follows that the vector $z - \frac{x+y}{2}$ is orthogonal to $\partial U \cap H$ at z . Since the normal to $\partial U \cap H$ at z is the orthogonal projection onto H of the normal to ∂U at z , this normal direction to ∂U at z lies in the plane spanned by $y - x$ and $z - \frac{x+y}{2}$. This implies that the normal line to ∂U through z is contained in the plane determined by the points x, y and z . \square

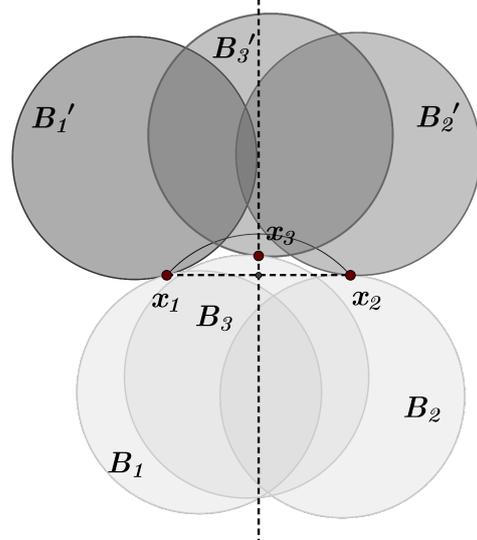


Fig. 3 A configuration of balls satisfying the Six Ball Lemma

Lemma 7 (Six Ball Lemma) Given three pairs of r -balls, (B_i, B'_i) , $i = 1, 2, 3$, such that (see Fig. 3):

- (a) B_i and B'_i are tangent at x_i , $i = 1, 2, 3$,
- (b) $(B_1 \cup B_2 \cup B_3) \cap (B'_1 \cup B'_2 \cup B'_3) = \emptyset$,
- (c) $0 < \|x_1 - x_2\| < 2r$,
- (d) $\|x_3 - x_1\| = \|x_3 - x_2\|$ and $\|x_3 - \frac{x_1+x_2}{2}\| \leq \frac{1}{2} \|x_1 - x_2\|$,
- (e) The line through the centres of B_3 and B'_3 is contained in the plane determined by the points x_1, x_2 and x_3 .

Then we have

$$\left\| x_3 - \frac{x_1 + x_2}{2} \right\| \leq r - \sqrt{r^2 - \frac{\|x_1 - x_2\|^2}{4}}.$$

Proof The proof goes by contradiction. Let $x_0 := \frac{x_1+x_2}{2}$ and $s := \|x_1 - x_2\|$, so that $\|x_3 - x_0\| \leq \frac{s}{2}$, but assume that $h := \|x_3 - x_0\| > r - \sqrt{r^2 - \frac{s^2}{4}}$. We shall derive a contradiction from this. Let $\eta = \frac{r}{\|x_3 - x_0\|}(x_0 - x_3)$ and $\eta' = \frac{r}{\|x_2 - x_1\|}(x_2 - x_1)$. These are orthogonal vectors with $\|\eta\| = \|\eta'\| = r$. Consider also the unique vector $\eta_3 \in \mathbb{R}^n$ such that (i) $\|\eta_3\| = r$, (ii) $x_3 + \eta_3$ is the centre of B_3 , and (iii) $x_3 - \eta_3$ is the centre of B'_3 . By assumption (e), the vector η_3 lies in the plane spanned by η and η' . We can assume that $\langle \eta, \eta_3 \rangle \geq 0$. Otherwise we work with $-\eta_3$ and B'_3 instead of η_3 and B_3 . Hence we can write $\eta_3 = (\cos \theta)\eta + (\sin \theta)\eta'$ with $\theta \in [-\frac{\pi}{2}, \frac{\pi}{2}]$. Exchanging the roles of x_1 and x_2 (if necessary) we can also assume that $\theta \in [0, \frac{\pi}{2}]$. Now, consider the function $\varphi : [0, \frac{\pi}{2}] \rightarrow \mathbb{R}$,

$$\begin{aligned} \varphi(\theta) &= \|x_3 + \eta_3 - x_2\|^2 = \|x_3 - x_2 + (\cos \theta)\eta + (\sin \theta)\eta'\|^2 \\ &= \left\| x_0 - x_2 - \frac{h}{r}\eta + (\cos \theta)\eta + (\sin \theta)\eta' \right\|^2 \\ &= \left\| \left(\cos \theta - \frac{h}{r} \right) \eta + \left(\sin \theta - \frac{s}{2r} \right) \eta' \right\|^2 \\ &= (r \cos \theta - h)^2 + \left(r \sin \theta - \frac{s}{2} \right)^2. \end{aligned}$$

Note that since $h > r - \sqrt{r^2 - \frac{s^2}{4}}$,

$$\varphi(0) = (r - h)^2 + (s/2)^2 < \left(r^2 - \frac{s^2}{4} \right) + \frac{s^2}{4} = r^2.$$

Also, since $h \leq s/2$ and $s < 2r$,

$$\varphi\left(\frac{\pi}{2}\right) = h^2 + (r - s/2)^2 \leq r^2 + s \left(\frac{s}{2} - r \right) < r^2.$$

It is not difficult to see that the function $\varphi(\theta)$ attains its minimum value at the argument of the vector $h\eta + \frac{s}{2}\eta'$ w.r.t. the orthonormal frame $\{\eta, \eta'\}$, and its maximum value at one of the boundary points $\theta = 0$ or $\theta = \frac{\pi}{2}$. Thus, from the previous inequalities it follows that $\|x_3 + \eta_3 - x_2\|^2 = \varphi(\theta) < r^2$. This shows that $x_2 \in B_3$, and hence B_3 intersects both B_2 and B'_2 , thus contradicting assumption (b). \square

Remark 2 For any $0 < s < 2r$,

$$r - \sqrt{r^2 - \frac{s^2}{4}} < \frac{s}{2}$$

with equality at the endpoints of $]0, 2r[$. Hence the conclusion of the Six Ball Lemma is an improvement of the a priori bound in assumption (d).

Define $\varphi : [0, 2r] \rightarrow [0, r]$,

$$\varphi(s) = r - \sqrt{r^2 - \frac{s^2}{4}}$$

and $\psi : [0, 2r] \rightarrow \mathbb{R}$,

$$\psi(s) = \sqrt{\frac{s^2}{4} + \varphi(s)^2} = \sqrt{2r^2 - 2r\sqrt{r^2 - \frac{s^2}{4}}}$$

(see Fig. 4).

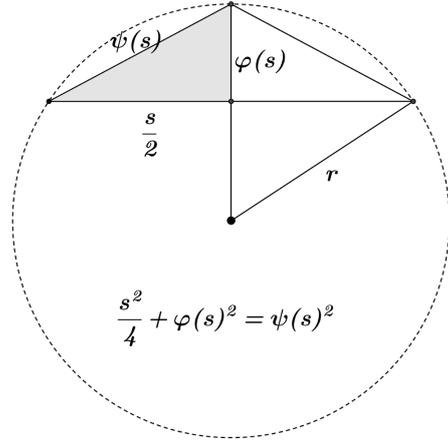


Fig. 4 Geometric meaning of $\psi(s)$ and $\varphi(s)$

Corollary 3 Let $U \subseteq \mathbb{R}^n$ be an r -regular set. Given $x, y \in \partial U$ such that $\|x - y\| < 2r$ there is $z \in \partial U$ satisfying

- (1) $\langle z - \frac{x+y}{2}, y - x \rangle = 0$,
- (2) $\left\| z - \frac{x+y}{2} \right\| \leq \varphi(\|x - y\|)$,
- (3) $\|z - x\| \leq \psi(\|x - y\|)$,
- (4) $\|z - y\| \leq \psi(\|x - y\|)$.

Proof Using Corollary 2, an application of the Six Ball Lemma gives (1) and (2). Pithagoras' Theorem implies (3) and (4). \square

Lemma 8 Given $s \in [0, 2r]$,

$$\lim_{n \rightarrow +\infty} 2^n \psi^n(s) = 2r \arctan \left(\frac{s}{\sqrt{4r^2 - s^2}} \right).$$

Proof (Geometric Proof) Consider a chord \overline{AB} of length $s < 2r$ in a circle \mathcal{C} of radius r . Let Γ denote the shortest arch of \mathcal{C} connecting the points A and B , which has length $2r \arctan \left(\frac{s}{\sqrt{4r^2 - s^2}} \right)$.

We can approximate Γ by a polygonal line Γ_n consisting of 2^n equal sides and all vertexes in \mathcal{C} . Recursively, we set $\Gamma_0 = \overline{AB}$ and define Γ_n to be the polygonal line obtained from Γ_{n-1} replacing each side \overline{XY} of Γ_{n-1} by the two equal sides \overline{XZ} and \overline{ZY} , where Z is the nearest point where the line that bisects \overline{XY} intersects the circle \mathcal{C} . It is easy to check, inductively in n , that $\text{length}(\Gamma_n) = 2^n \psi^n(s)$ for every $n \geq 0$. Thus

$$\begin{aligned} \lim_{n \rightarrow +\infty} 2^n \psi^n(s) &= \lim_{n \rightarrow +\infty} \text{length}(\Gamma_n) = \text{length}(\Gamma) \\ &= 2r \arctan\left(\frac{s}{\sqrt{4r^2 - s^2}}\right). \end{aligned}$$

□

The following theorem proves Proposition 9.

Theorem 4 *Let $U \subseteq \mathbb{R}^n$ be an r -regular set. Given $x, y \in \partial U$ such that $\|x - y\| < 2r$, the intrinsic distance between x and y in ∂U satisfies*

$$d_{\partial U}(x, y) \leq 2r \arctan\left(\frac{\|x - y\|}{\sqrt{4r^2 - \|x - y\|^2}}\right) < \pi r.$$

Proof Applying Corollary 3 inductively we can construct a sequence of polygonal lines Γ_n with 2^n equal sides, and all vertexes in ∂U . By construction the polygonal line Γ_n has length $\leq 2^n \psi^n(\|x - y\|)$, and can be parametrized as a Lipschitz curve with Lipschitz constant $1/r$. The sequence Γ_n is a Cauchy sequence w.r.t. $\|\cdot\|_\infty$. Therefore in view of Lemma 8 $\Gamma = \lim_{n \rightarrow \infty} \Gamma_n$ is a Lipschitz curve connecting x to y with length

$$\text{length}(\Gamma) \leq 2r \arctan\left(\frac{\|x - y\|}{\sqrt{4r^2 - \|x - y\|^2}}\right).$$

□

Together, Proposition 2, Proposition 8 and Proposition 9 prove Theorem 3.

10 Conclusions and Future Work

In this paper we have provided a characterization of r -regular sets in terms of the Lipschitz regularity of normal vector fields to the boundary.

We would like to consider a possible extension of the notion of r -regular sets regarding its application to the problem of surface reconstruction. We also plan to pursue our research on smooth non-deterministic dynamical systems where the ‘smoothness’ of the boundary of r -regular sets will be applied.

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