Note on the ergodicity of the Gauss map

Abstract

This note sketches the proof of the ergodicity of the Gauss map.

Let $T : [0, 1] \to [0, 1]$ be the Gauss map, $T(x) = \frac{1}{x} - \lfloor \frac{1}{x} \rfloor$, and μ be the Gauss measure, which is defined by

$$\mu(A) = \frac{1}{\log 2} \int_A \frac{1}{1+x}\, dx \ .$$

The Gauss map with this measure determines a measure-preserving dynamical system (MPDS) $(T, [0, 1], \mathcal{F}, \mu)$.

Theorem 1. *The MPDS (T, μ) is ergodic.*

Given two functions $f_1, f_2 : X \to \mathbb{R}$ we write $f_1(x) \asymp f_2(x)$ $(x \in X)$ to express that there exists a constant $c > 0$ such that for all $x \in X$,

$$e^{-c}\, f_2(x) \leq f_1(x) \leq e^{c}\, f_2(x).$$

With this terminology we will prove that

Proposition 1. $\mu(A \cap T^{-n} B) \asymp \mu(A)\, \mu(B)$ $(A, B \in \mathcal{F}, n \in \mathbb{N})$.

Proof of Theorem 1. Given $B \in \mathcal{F}$ such that $T^{-1}B = B$, defining $A = X \setminus B$, we have

$$0 = \mu(\emptyset) = \mu(A \cap B) = \mu(A \cap T^{-n} B) \asymp \mu(A)\, \mu(B)$$

Hence $\mu(A) = 0$ or $\mu(B) = 0$, which implies that $\mu(B) = 1$ or $\mu(B) = 0$. Thus (T, μ) is ergodic. □

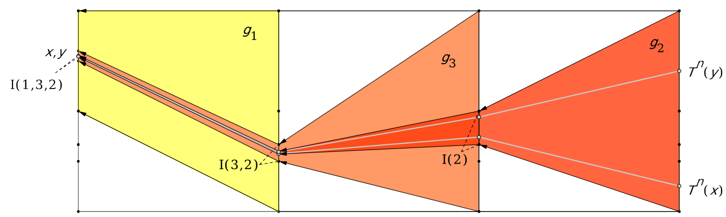
The rest of this note is devoted to prove the previous proposition.

Given integers $a_0, a_1, \ldots, a_n \geq 1$ define

$$I(a_1, \ldots, a_n) := \{\, [a_1, \ldots, a_n, x] \,:\, x \in [0, 1]\,\}.$$

Proposition 2. *$\exists\, c>0\ \forall\, a_1,\dots,a_n\in\mathbb{N}$, $T^n : I(a_1,\dots,a_n)\to[0,1]$ é um difeomorfismo tal que $\forall\, x,y\in I(a_1,\dots,a_n)$,*

$$e^{-c}\le\frac{|(T^n)'(x)|}{|(T^n)'(y)|}\le e^c\ .$$



Proof. $$(\log|T'(x)|)'=\frac{|T''(x)|}{|T'(x)|}=\frac{2}{|x|}\le 2$$

$$\log|(T^n)'(x)|=\sum_{i=1}^n\log\left|T'(T^{i-1}x)\right|$$

$$\begin{aligned}\left|\log\frac{|(T^n)'(x)|}{|(T^n)'(y)|}\right| &\le\sum_{i=1}^n\left|\log\left|T'(T^{i-1}x)\right|-\log\left|T'(T^{i-1}y)\right|\right|\\ &\le 2\sum_{i=1}^n\left|T^{i-1}(x)-T^{i-1}(y)\right|\\ &\le 2\sum_{i=1}^n\left|g_{a_{i-1}}\circ\ldots\circ g_{a_n}(T^nx)-g_{a_{i-1}}\circ\ldots\circ g_{a_n}(T^ny)\right|\\ &\le 2\sum_{i=1}^n\frac{1}{2^{n-i-1}}\le 8\ .\end{aligned}$$

$$I_n=I(a_1,\dots,a_n)=\{\,g_{a_1}\circ\ldots\circ g_{a_n}(x)\ :\ 0\le x\le 1\,\}$$

$\forall\,[\alpha,\beta]\subseteq[0,1]$,
$I_n\cap T^{-n}[\alpha,\beta]=\{\,g_{a_1}\circ\ldots\circ g_{a_n}(x)\ :\ \alpha\le x\le\beta\,\}$

$$\beta-\alpha=m[\alpha,\beta]=|(T^n)'(x)|\ m(I_n\cap T^{-n}[\alpha,\beta])$$
$$1=m[0,1]=|(T^n)'(y)|\ m(I_n)$$
$$\Downarrow$$
$$m(I_n\cap T^{-n}[\alpha,\beta])=m[\alpha,\beta]\,m(I_n)\,\tfrac{|(T^n)'(y)|}{|(T^n)'(x)|}$$
$$\Downarrow$$
$$e^{-c}\,m[\alpha,\beta]\,m(I_n)\le m(I_n\cap T^{-n}[\alpha,\beta])\le e^c\,m[\alpha,\beta]\,m(I_n)$$

$\forall\, J=I(a_1,\dots,a_k)$ com $k<n$, J é uma união numerável de intervalos disjuntos $I_n(a_1,\dots,a_k,b_{k+1},\dots,b_n)$, $b_{k+1},\dots,b_n\in\mathbb{N}$.

$$\Downarrow$$

$$\forall\, [\alpha,\beta] \subseteq [0,1],$$
$$e^{-c}\, m[\alpha,\beta]\, m(J) \le m(J \cap T^{-n}[\alpha,\beta]) \le e^{c}\, m[\alpha,\beta]\, m(J)$$

$$\Downarrow$$

$$m(J \cap T^{-n}[\alpha,\beta]) \asymp m[\alpha,\beta]\, m(J) \quad (J, [\alpha,\beta])$$

$$\Downarrow$$

$$m(A \cap T^{-n}B) \asymp m(A)\, m(B) \quad (A, B \in \mathcal{A})$$

□

Lemma 1. *Let m denote the Lebesgue measure on $[0,1]$. Then*

(a) $\mu(A) \asymp m(A)$ $(A \in \mathcal{F})$

(b) $m(A \cap T^{-n}B) \asymp m(A)\, m(B)$ $(A, B \in \mathcal{F})$

(a) $\quad \frac{1}{2\,\log 2}\, m(A) \le \mu(A) = \frac{1}{\log 2} \int_A \frac{1}{1+x}\, dx \le \frac{1}{\log 2}\, m(A)$

(a) + (b) $\quad \Rightarrow\ \mu(A \cap T^{-n}B) \asymp \mu(A)\, \mu(B)$ $(A, B \in \mathcal{A})$

Theorem 2. *For almost every $x \in [0,1]$, the convergents*

$$\frac{p_n(x)}{q_n(x)} = [a_0(x), a_1(x), \ldots, a_{n-1}(x)]$$

of x satisfy

$$\lim_{n\to+\infty} \frac{1}{n}\,(\, a_0(x) + \ldots + a_{n-1}(x)\,) = \infty$$

$$\lim_{n\to+\infty} \frac{1}{n}\, \log q_n(x) = \frac{\pi^2}{12\,\log 2} = 1.186\ldots$$

$$\lim_{n\to+\infty} \frac{1}{n}\, \log \left| x - \frac{p_n(x)}{q_n(x)} \right| = -\frac{\pi^2}{6\,\log 2} = -2.373\ldots$$