ON THE INTEGRAL REPRESENTATION OF VARIATIONAL FUNCTIONALS ON BD[*]

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Abstract. Following the global method for relaxation we prove an integral representation result for a large class of variational functionals naturally defined on the space of functions with bounded deformation. Mild additional continuity assumptions are required on the functionals.

1. Introduction. In linearized elasticity one route is to consider the displacement (or the velocity) field u as basic model variable. In this case, the deformation (or strain) tensor of an elastic body is given by the symmetric gradient of u, i.e., $e(u) := \frac{1}{2}(\nabla u + \nabla^t u)$. Therefore, the study of well-posedness of the PDE system of linear elasticity was at the origin of the study of the differential operator $e(u)$ and in particular its coerciveness properties, first analyzed by Korn in 1906 [34] and followed by plenty of refinements to this date (see, for instance, [30] for a survey). In linearized elasticity, the variational approach consists in minimizing the stored elastic energy (which is quadratic in the strain) minus the work of the external forces. However, as soon as elasto-plasticity is considered, two main problems are faced. First, the observed stress-strain relation in plasticity is not linear anymore, resulting in a less-than-quadratic, sometimes linear relation between the stored elastic energy and the strain. Here we refer to the pioneer work by Suquet on well-posedness in perfect plasticity [46], itself based on preliminary work on the distributional operator of bounded deformation published in [45, 44, 49, 50, 33]. Specifically, in the above quoted works the authors study the properties of the differential operator $Eu := \frac{1}{2}(Du + D^t u)$, where D stands for the distributional derivative that generalizes the gradient ∇ to account for discontinuous fields u. In this way the space $BD(\Omega)$ of a function with bounded deformation on the open subset Ω of $\mathbb{R}^n$ has been introduced as the space of $L^1(\Omega; \mathbb{R}^n)$ vector fields u whose symmetrized distributional derivative Eu is a Radon measure (see [48, 1]; see also section 3.2, to which we refer for the notation used in this introduction on BD maps). Moreover, $e(u)$ is proven to be the density of the absolutely continuous part of Eu.

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The second issue arising in plastic problems is that concentration phenomena observed in plasticity require some weak notion of deformation that allow for slip or boundary concentration of strain, for instance. Indeed, these effects are well handled in $BD(\Omega)$ by the so-called singular part of the deformation measure field. It should also be said that these aforementioned two issues are related, since a linear growth of the stored elastic energy prevents coerciveness in Sobolev spaces. Thus, bounds in the nonreflexive space L^1 require considering limit of sequences in the space of Radon measures, and hence, again, justify the choice of the space $BD(\Omega)$ when dealing with elasto-plastic models. For these models, the associated general bulk stored elastic energy reads as the integral

$$(1.1) \qquad F_0(u) := \int_\Omega f_0\big(x, u(x), e(u)(x)\big)\,\mathrm{d}x,$$

where f_0 has linear growth in the third variable and satisfies suitable assumptions (see section 6.1), and $u \in BD(\Omega)$ is such that Eu is absolutely continuous with respect to $\mathcal{L}^n \llcorner \Omega$, namely $u \in LD(\Omega)$. To account also for singular effects a more general energy expression reads as

$$(1.2) \quad F_1(u) := \int_\Omega f_1\big(x, u(x), e(u)(x)\big)\,\mathrm{d}x + \int_{J_u} g_1\big(x, u^-(x), u^+(x), \nu_u(x)\big)\,d\mathcal{H}^{n-1},$$

where f_1 has linear growth in the third variable, f_1 and g_1 satisfy suitable assumptions (see section 6.2), and $u \in SBD(\Omega)$, the subspace of $BD(\Omega)$ where the singular part $E^s u$ of the measure Eu is concentrated on the $(n-1)$-rectifiable set of approximate discontinuity points J_u (see section 3.2 for the precise notation).

It is a classical problem in the calculus of variations to determine the lower semicontinuous envelope of the energies in (1.1) and (1.2) in order to find the limits of minimizing sequences lying in the larger space $BD(\Omega)$. More precisely, let F be the functional either in (1.1) or in (1.2) if u belongs to $LD(\Omega)$ or $SBD(\Omega)$, respectively, and $+\infty$ otherwise on $L^1(\Omega; \mathbb{R}^n)$. Then, the $L^1(\Omega; \mathbb{R}^n)$ lower semicontinuous envelope of the functional F, that is, the greatest functional less than or equal to F which is L^1 lower semicontinuous, is given by

$$\mathcal{F}(u) := \inf\left\{\liminf_{j \to +\infty} F(u_j) : u_j \to u \text{ in } L^1(\Omega; \mathbb{R}^n)\right\},$$

provided some coercivity assumptions on the integrands are imposed (cf. [27]).

A suitable localized version of the functional $\mathcal{F}$ to the family of open subsets of Ω turns out to be a variational functional according to Dal Maso and Modica [17] naturally defined on the space $BD(\Omega)$ (see section 2.2 below and the discussion in what follows). Therefore, more generally, we consider variational functionals in this sense and prove for them in Theorem 2.3 an integral representation result following closely the celebrated global method for relaxation developed in Bouchitté, Fonseca, and Mascarenhas [7] to deal with the analogous problem for functionals defined either on Sobolev spaces or on the space BV of functions with bounded variation (see [14] for an extensive survey of the subject and an exhaustive bibliography).

The integral representation result in Theorem 2.3 will be applied to the above mentioned relaxation problems for the functionals of the type (1.1) and (1.2). More precisely, in Theorems 6.1 and 6.9 we show that the resulting L^1 lower semicontinuous envelope $\mathcal{F}$ has indeed an integral representation of the type

$$(1.3) \qquad \mathcal{F}(u) = \int_\Omega f\big(x, u(x), e(u)(x)\big)\, \mathrm{d}x$$

$$+ \int_{J_u} g\big(x, u^-(x), u^+(x), \nu_u(x)\big)\, \mathrm{d}\mathcal{H}^{n-1}(x)$$

$$+ \int_\Omega f^\infty \left(x, u(x), \frac{\mathrm{d}E^c u}{\mathrm{d}|E^c u|}(x)\right)\, \mathrm{d}|E^c u|$$

for all $u \in BD(\Omega)$. Here, f^∞ denotes the (weak) recession function of the integrand f. Moreover, a characterization of the energy densities f and g is given in terms of asymptotic Dirichlet problems involving F itself with boundary values related to the infinitesimal behaviors of the function u around the base point x.

Apart from the usual lower semicontinuity (and therefore locality), growth conditions, and measure theoretical properties to be satisfied by the functional $\mathcal{F}$ (see assumptions (H1)–(H3) in section 2), we impose two conditions expressing continuity of the energy functional with respect to a specific family of rigid motions. More precisely, continuity with respect to translations in both the dependent and the independent variable is stated in (H4). Such a condition is used, for instance, in [7] in the BV setting to express the energy density of the Cantor part in terms of the recession function f^∞ of the bulk energy density. Additionally, in the current BD setting we need to require further assumption (H5), which expresses continuity of the energy with respect to infinitesimal rigid motions. In turn, this condition implies that the bulk energy density depends only on the symmetric part of the relevant matrix. Condition (H5) is crucial for our arguments both from a technical side and conceptually as we discuss in detail in section 7.

In this respect, we emphasize that all integral representation, relaxation, and lower semicontinuity results available in the literature for energies defined on $BD(\Omega)$ (see, e.g., [5, 24, 42, 22, 4, 35, 18]) are based on a stronger version of (H5) that imposes invariance of the energy with respect to infinitesimal rigid motions (cf. Remark 5.2). From a mechanical perspective such a condition reflects a restriction on the material behavior. Therefore, it is preferable to avoid it, also because of its controversy in the continuum mechanics community (see [47, 40]). Note that the quoted invariance property with respect to superposed infinitesimal rigid motions would imply the integrands in (1.3) to be independent of u. In our result, though, this explicit dependence is kept, as was the case in the $BV(\Omega)$ setting [7]. Finally, let us point out that assumption (H5) is actually not needed to give a partial integral representation result on the subspace $SBD(\Omega)$ or more generally on $BD(\Omega)$ but only for the volume and surface terms of the energies, as already noticed in [24].

Further possible applications of our main theorem are in the field of homogenization problems (cf. section 6.3), or more generally to problems in which the determination of variational limits in terms of Γ-convergence of energies defined on BD are involved (see, e.g., [16], [26, 13, 11]).

We mention that integral representation results for energies defined on distinguished subspaces of BD (in particular satisfying a set of growth conditions different from (H2)) have been recently obtained either in the superlinear case in the two-dimensional framework in [14] or in the space of Caccioppoli affine functions in [29].

Let us now summarize the contents of the paper. In section 2 we state Theorem 2.3, the main result of the paper, and all the preliminaries needed to prove it are provided in section 3. Section 4 focuses on the analysis of the Cantor part of the energy and more precisely on its integral representation. In turn, those results are used in section 5 to establish Theorem 2.3. Applications of Theorem 2.3 to several

issues related to energies with linear growth defined on *BD* are studied in section 6. More precisely, the relaxation of variational integrals is the topic of section 6.1, the lower semicontinuity either of bulk or of bulk and surface energies is investigated in section 6.2, then section 6.3 deals with the periodic homogenization of bulk type energies. In the final section 7 an example of a quasiconvex function $f : \mathbb{M}^{n \times n} \to [0, +\infty)$ being bounded by the norm of the symmetric part of the relevant matrix, but on the other hand depending nontrivially also on the skew-symmetric part, is provided following a celebrated example by Müller [39, Theorem 1]. The issue of relaxation on *BD* for the associated bulk energy functional is discussed, highlighting the role of assumption (H5) to deduce Theorem 2.3 and related open problems.

2. Main result.

2.1. Basic notation.
The unitary vectors of the standard coordinate basis of $\mathbb{R}^n$ will be denoted by $\mathrm{e}_1, \ldots, \mathrm{e}_n$. $\mathbb{M}^{n \times n}$ stands for the set of all $n \times n$ matrices and $\mathbb{M}^{n \times n}_{sym}$, $\mathbb{M}^{n \times n}_{skew}$ for the subsets of all symmetric and skew-symmetric matrices, respectively.

For a given a set E we adopt the notation $E(x_0, r) := x_0 + rE$ for the rescaled copy of size $r > 0$ translated in x_0. In particular, $Q^\nu(x_0, r)$ stands for any cube centered at x_0, with edge length r and with one face orthogonal to ν. We also adopt the convention that whenever ν is omitted, then $\nu = \mathrm{e}_i$ for some $i \in \{1, \ldots, n\}$, and the corresponding cube $Q(x_0, r)$ is oriented according to the coordinate directions.

In what follows, we shall often consider a function $\Psi : (0, +\infty) \to [0, +\infty)$ such that $\lim_{t \to 0^+} \Psi(t) = 0$, referring to it as a modulus of continuity.

Finally, throughout the paper Ω will denote a nonempty, bounded, open subset of $\mathbb{R}^n$ with Lipschitz boundary, and $\mathcal{O}(\Omega)$, $\mathcal{O}_\infty(\Omega)$ denote the families of all the open subsets of Ω and of all the open subsets of Ω with Lipschitz boundary, respectively.

2.2. Framework and main result.
We consider a class of local energies typically arising in variational problems, $\mathcal{F} : L^1(\Omega; \mathbb{R}^n) \times \mathcal{O}(\Omega) \to [0, +\infty]$, and $BD(\Omega)$ is the set of maps with *bounded deformation* (for the precise definition, the notation used in what follows, and several properties of *BD* functions see section 3). We assume that the following properties are in force on $\mathcal{F}$:

(H1) $\mathcal{F}(\cdot, A)$ is strongly $L^1(A; \mathbb{R}^n)$ lower semicontinuous for all $A \in \mathcal{O}(\Omega)$.[1]

(H2) There exists a constant $C > 0$ such that for every $(u, A) \in BD(\Omega) \times \mathcal{O}(\Omega)$,

$$(2.1) \qquad \frac{1}{C}|Eu|(A) \leq \mathcal{F}(u, A) \leq C(\mathcal{L}^n(A) + |Eu|(A)).$$

(H3) $\mathcal{F}(u, \cdot)$ is the restriction to $\mathcal{O}(\Omega)$ of a Radon measure for every $u \in BD(\Omega)$.

(H4) There exists a modulus of continuity Ψ such that

$$(2.2)$$
$$|\mathcal{F}(\mathrm{v} + u(\cdot - x_0), x_0 + A) - \mathcal{F}(u, A)| \leq \Psi(|x_0| + |\mathrm{v}|)(\mathcal{L}^n(A) + |Eu|(A))$$

for all $(u, A, \mathrm{v}, x_0) \in BD(\Omega) \times \mathcal{O}(\Omega) \times \mathbb{R}^n \times \Omega$, with $x_0 + A \subset \Omega$.

(H5) There exists a modulus of continuity Ψ such that

$$(2.3) \quad |\mathcal{F}(u + \mathbb{L}(\cdot - x_0), A) - \mathcal{F}(u, A)| \leq \Psi(|\mathbb{L}|\mathrm{diam}(A))(\mathcal{L}^n(A) + |Eu|(A))$$

for every $(u, A, \mathbb{L}) \in BD(\Omega) \times \mathcal{O}(\Omega) \times \mathbb{M}^{n \times n}_{skew}$, and for all $x_0 \in A$.

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[1] In the rest of the paper, we shall write L^1 lower semicontinuous in place of strongly L^1 lower semicontinuous for the sake of simplicity.

Remark 2.1. It is well-known that assumption (H1) implies locality of $\mathcal{F}(\cdot, A)$ for all $A \in \mathcal{O}(\Omega)$. Namely, if $u = v$ $\mathcal{L}^n$ a.e. on A, then $\mathcal{F}(u, A) = \mathcal{F}(v, A)$.

Remark 2.2. Hypothesis (H5) implies that the energy depends only on the symmetric gradient (see formula (5.2) and section 7 for more details).

Following the global method for relaxation introduced by Bouchitté, Fonseca, and Mascarenhas in [7] we consider the *local* Dirichlet problem

$$(2.4) \qquad \mathfrak{m}(u, A) := \inf \left\{ \mathcal{F}(v, A) : v \in BD(\Omega), \ v|_{\partial A} = u|_{\partial A} \right\},$$

where $(u, A) \in BD(\Omega) \times \mathcal{O}_\infty(\Omega)$ is given, and prove the ensuing result.

THEOREM 2.3. *Let* $\mathcal{F} : L^1(\Omega; \mathbb{R}^n) \times \mathcal{O}(\Omega) \to [0, +\infty]$ *satisfy* (H1)–(H5). *Then, for all* $(u, A) \in BD(\Omega) \times \mathcal{O}(\Omega)$

$$\mathcal{F}(u, A) = \int_A f\big(x, u(x), e(u)(x)\big)\, \mathrm{d}x + \int_{J_u \cap A} g\big(x, u^-(x), u^+(x), \nu_u(x)\big)\, \mathrm{d}\mathcal{H}^{n-1}(x)$$
$$+ \int_A f^\infty \left(x, u(x), \frac{\mathrm{d}E^c u}{\mathrm{d}|E^c u|}(x) \right) \mathrm{d}|E^c u|(x),$$

where for all $(x_0, \mathbb{A}, \mathrm{v}, \mathrm{v}^-, \mathrm{v}^+, \nu) \in \Omega \times \mathbb{M}_{sym}^{n \times n} \times (\mathbb{R}^n)^4$

$$(2.5) \qquad f(x_0, \mathrm{v}, \mathbb{A}) := \limsup_{\varepsilon \to 0} \frac{\mathfrak{m}(\mathrm{v} + \mathbb{A}(\cdot - x_0), Q(x_0, \varepsilon))}{\varepsilon^n},$$

$$(2.6) \qquad g(x_0, \mathrm{v}^-, \mathrm{v}^+, \nu) := \limsup_{\varepsilon \to 0} \frac{\mathfrak{m}(u_{\mathrm{v}^-, \mathrm{v}^+, \nu}(\cdot - x_0), Q^\nu(x_0, \varepsilon))}{\varepsilon^{n-1}},$$

$$(2.7) \qquad f^\infty(x_0, \mathrm{v}, \mathbb{A}) := \limsup_{t \to +\infty} \frac{f(x_0, \mathrm{v}, t\mathbb{A}) - f(x_0, \mathrm{v}, 0)}{t},$$

and

$$(2.8) \qquad u_{\mathrm{v}^-, \mathrm{v}^+, \nu}(y) := \left\{ \begin{array}{ll} \mathrm{v}^+ & \textit{if } y \cdot \nu \geq 0, \\ \mathrm{v}^- & \textit{otherwise.} \end{array} \right.$$

Note that f^∞ is classically termed the *weak* recession function, in contrast to the *strong* recession function for which the limit is assumed to exist (see the comments in section 6.1). The finiteness of f^∞ is guaranteed by the linear growth of f (see (5.1)).

We point out that the analogous result to Theorem 2.3 for functionals defined on the space BV of functions with bounded variation has been proven under the sole assumptions (H1)–(H4) (cf. [7, Theorems 3.7, 3.12]). A detailed discussion on the need of assumption (H5) in the BD setting is the topic of section 7. Several comparisons with the BV case are discussed in Remarks 6.2 and 6.10.

3. Preliminaries.

3.1. Some results of geometric measure theory. In the forthcoming blow-up procedure, it will be mandatory to obtain limits satisfying additional structural properties. To this aim we introduce some useful concepts of geometric measure theory. Let $U \subseteq \mathbb{R}^n$ be either open or closed. Here and in what follows $\mathcal{M}_{loc}(U; \mathbb{R}^k)$ stands for the sets of all $\mathbb{R}^k$-valued Radon measures on U, and $\mathcal{M}(U; \mathbb{R}^k)$ for the subset of all $\mathbb{R}^k$-valued finite Radon measures on U.

Let μ_i, $\mu \in \mathcal{M}_{loc}(U;\mathbb{R}^k)$; following [3] we say that μ_i locally weakly* converge to μ, and we write $\mu_i \overset{*}{\rightharpoonup} \mu$ in $\mathcal{M}_{loc}(U;\mathbb{R}^k)$, if

$$\lim_{i \to +\infty} \int_U \varphi \,\mathrm{d}\mu_i = \int_U \varphi \,\mathrm{d}\mu \qquad \text{for all } \varphi \in \mathcal{C}_c^0(U).$$

If μ_i, $\mu \in \mathcal{M}(U;\mathbb{R}^k)$, we say that μ_i weakly* converges to μ if the condition above holds for all $\varphi \in \mathcal{C}_0^0(U)$. The following property holds true.

LEMMA 3.1 (see [3, Proposition 1.62(b)]). *Let $\{\mu_i\}_{i\in\mathbb{N}}$ be locally weakly* converging to μ in $\mathcal{M}_{loc}(U;\mathbb{R}^k)$, and let $\{|\mu_i|\}_{i\in\mathbb{N}}$ be locally weakly* converging to λ in $\mathcal{M}_{loc}(U)$. Then, $\lambda \geq |\mu|$, and $\mu(E) = \lim_{i\to\infty} \mu_i(E)$ for every relatively compact Borel subset E of U with $|\lambda|(\partial E) = 0$.*

We use standard notation for the push-forward of measures, and in particular, given $\mu \in \mathcal{M}_{loc}(\mathbb{R}^n;\mathbb{R}^k)$, $x \in \mathbb{R}^n$, and $\varepsilon > 0$, we will often consider the push-forward with the map $F^{x,\varepsilon}(y) := \frac{y-x}{\varepsilon}$ defined as

$$(3.1) \qquad\qquad F_\#^{x,\varepsilon}\mu(A) := \mu(x + \varepsilon A).$$

Preiss' tangent space $\mathrm{Tan}(\mu,x)$ at a given point $x \in \mathbb{R}^n$ is defined as the subset of nonzero measures $\nu \in \mathcal{M}_{loc}(\mathbb{R}^n;\mathbb{R}^k)$ such that ν is the local weak* limit of $^1/_{c_i}F_\#^{x,\varepsilon_i}\mu$, for some sequence $\{\varepsilon_i\}_{i\in\mathbb{N}} \downarrow 0$ as $i \uparrow +\infty$ and for some positive sequence $\{c_i\}_{i\in\mathbb{N}}$ (see [38], [3], [43]). To ensure that the total variation is preserved along the blow-up limit procedure we recall the ensuing result.

LEMMA 3.2 (tangent measure with unit mass [43, Lemma 10.6]). *Let $\mu \in \mathcal{M}_{loc}(\mathbb{R}^n;\mathbb{R}^k)$. Then, for $|\mu|$-a.e. $x \in \mathbb{R}^n$ and for every bounded, open, convex set K the following assertions hold:*

(a) *There exists a tangent measure $\gamma \in \mathrm{Tan}(\mu,x)$ such that $|\gamma|(K) = 1$, $|\gamma|(\partial K) = 0$.*

(b) *There exists $\{\varepsilon_i\}_{i\in\mathbb{N}} \downarrow 0$ as $i \uparrow +\infty$ such that $\frac{F_\#^{x,\varepsilon_i}\mu}{F_\#^{x,\varepsilon_i}|\mu|(K)} \overset{*}{\rightharpoonup} \gamma$ in $\mathcal{M}(\overline{K};\mathbb{R}^k)$.*

Finally, with the help of the next result, we will be able to select a blow-up with a partial affine structure.

THEOREM 3.3 (tangent measures to tangent measures are tangent measures; Theorem 14.16 in [38]). *Let $\mu \in \mathcal{M}_{loc}(\mathbb{R}^n;\mathbb{R}^k)$ be a Radon measure. Then for $|\mu|$-a.e. $x \in \mathbb{R}^n$ any $\nu \in \mathrm{Tan}(\mu,x)$ satisfies the following properties:*

(a) *For any convex set K, $\frac{F_\#^{y,\rho}\nu}{F_\#^{y,\rho}|\nu|(K)} \in \mathrm{Tan}(\mu,x)$ for all $y \in \mathrm{spt}\,\nu$ and $\rho > 0$.*

(b) *$\mathrm{Tan}(\nu,y) \subseteq \mathrm{Tan}(\mu,x)$ for all $y \in \mathrm{spt}\,\nu$.*

Note that the original result in [38, Theorem 14.16] is proven for $k = 1$. However, the good properties of tangent space of measures (i.e., [3, Theorem 2.44] or [43, Lemma 10.4]) allow us to immediately extend its validity for generic k.

3.2. Preliminaries on BD. We recall next some basic properties of the space BD needed for our purposes. We refer to [48] for classical theorems, while for the fine properties we refer to [1] (see also [23]).

The space of functions with bounded deformation on Ω, $BD(\Omega)$, is the set of all maps $u \in L^1(\Omega;\mathbb{R}^n)$ whose symmetrized distributional derivative Eu is a matrix-valued Radon measure. It is a Banach space equipped with the norm $\|u\|_{BD(\Omega)} := \|u\|_{L^1(\Omega,\mathbb{R}^n)} + |Eu|(\Omega)$, where $|\mu|$ stands for the total variation of the Radon measure μ (see [3]). A sequence $\{u_j\}_{j\in\mathbb{N}}$ is said to strictly converge to u in $BD(\Omega)$ if

$$u_j \to u \text{ in } L^1(\Omega; \mathbb{R}^n) \text{ and } |Eu_j|(\Omega) \to |Eu|(\Omega),$$

as $j \to \infty$.

As shown by Ambrosio, Coscia, and Dal Maso in [1], $BD(\Omega)$ maps are approximately differentiable $\mathcal{L}^n$-a.e. in Ω, the jump set is $\mathcal{H}^{n-1}$-rectifiable, and Eu can be decomposed as

$$(3.2) \qquad Eu = e(u)\mathcal{L}^n \llcorner \Omega + (u^+(x) - u^-(x)) \odot \nu_u \mathcal{H}^{n-1} \llcorner J_u + E^c u,$$

where $e(u) = \frac{\nabla u + \nabla u^t}{2}$, ∇u is the approximate gradient of u, $u^+ - u^-$ denotes the jump of u over the jump set J_u, with $u^\pm$ the traces left by u on J_u, ν_u is a unitary Borel vector field normal to J_u (here, $a \odot b := \frac{1}{2}(a \otimes b + b \otimes a)$, a, $b \in \mathbb{R}^n$, denotes the symmetrized tensor product), $E^c u$ is the Cantor part of Eu defined as $E^c u := E^s u \llcorner (\Omega \setminus J_u)$, and $E^s u := Eu - e(u)\mathcal{L}^n \llcorner \Omega$ (cf. [1, equation (1.2), Definition 4.1]). Let S_u be the complement of the set of points of approximate continuity of u; then [1, Theorem 6.1] implies that $|Eu|(S_u \setminus J_u) = 0$, so that $E^c u = Eu \llcorner C_u$, where

$$(3.3) \qquad C_u := \left\{ x \in \Omega \setminus S_u : \lim_{r \downarrow 0} \frac{|Eu|(B_r(x))}{r^n} = +\infty, \lim_{r \downarrow 0} \frac{|Eu|(B_r(x))}{r^{n-1}} = 0 \right\}.$$

The limits in the definition of C_u can be taken with respect to any family $K(x, r)$, with K a bounded, open, convex set containing the origin. We shall often use the previous characterization of $E^c u$ throughout the paper.

The space of special functions of bounded deformation is then defined as

$$SBD(\Omega) = \{u \in BD(\Omega) : E^c u = 0\}.$$

The space $\mathcal{C}^\infty(\Omega; \mathbb{R}^n) \cap W^{1,1}(\Omega; \mathbb{R}^n)$ is dense in $BD(\Omega)$ for the strict topology on $BD(\Omega)$. Moreover, for Ω an open bounded set with Lipschitz boundary, there exists a surjective, bounded, linear trace operator $\gamma : BD(\Omega) \longrightarrow L^1(\partial\Omega, \mathbb{R}^n)$ satisfying the following integration by parts formula: for every $u \in BD(\Omega)$ and $\varphi \in \mathcal{C}^1(\bar{\Omega})$,

$$\int_\Omega u \odot \nabla\varphi \, dx + \int_\Omega \varphi \, dEu = \int_{\partial\Omega} \varphi \, \gamma(u) \odot \nu \, d\mathcal{H}^{n-1},$$

with ν the unit external normal to $\partial\Omega$. The trace operator is continuous if $BD(\Omega)$ is endowed with the strict topology. For notational simplicity, in what follows $\gamma(u)$ will be denoted simply by u itself.

With the same assumptions on Ω, one also has the following embedding result: $BD(\Omega) \hookrightarrow L^q(\Omega, \mathbb{R}^n)$ is compact for every $1 \leq q < \frac{n}{n-1}$. In view of compactness, the following holds for Ω a bounded extension domain (cf. [33], [48]): if $\{u_j\}_{j \in \mathbb{N}}$ is bounded in $BD(\Omega)$ there exists a subsequence that converges to some $u \in BD(\Omega)$ with respect to the $L^1(\Omega; \mathbb{R}^n)$ topology.

We recall next a Poincaré inequality for BD maps which has been proven in [1, Theorem 3.1] (see also [23, Theorem 1.7.11] and also [23, Lemma 1.4.1]). To this aim consider the space of infinitesimal rigid motions

$$(3.4) \qquad \mathcal{R} := \left\{ \mathbb{L}x + \mathrm{v} \, : \, \mathrm{v} \in \mathbb{R}^n, \, \mathbb{L} \in \mathbb{M}^{n \times n}_{skew} \right\}.$$

THEOREM 3.4. *Let A be a bounded, open, connected set with Lipschitz boundary; then*

$$\mathcal{R} = \{\vartheta \in \mathcal{D}'(A; \mathbb{R}^n) : E\vartheta = 0\}. \tag{3.5}$$

Moreover, let $\mathfrak{R} : BD(A) \to \mathcal{R}$ be a linear continuous map which leaves the elements of $\mathcal{R}$ fixed. Then there exists a constant $c = c(A, \mathfrak{R})$ such that for all $u \in BD(A)$

$$\|u - \mathfrak{R}[u]\|_{L^{\frac{n}{n-1}}(A; \mathbb{R}^n)} \leq c|Eu|(A).$$

In particular, for a bounded, open convex set K, denoting by $\nu_{\partial K}$ the unit normal vector to ∂K, let $\mathfrak{R}_K : BD(K) \to \mathcal{R}$ be the map defined as

$$\mathfrak{R}_K[u](y) := \mathbb{M}_K[u]y + b_K[u], \tag{3.6}$$

with

$$b_K[u] := \fint_K u \, dx, \tag{3.7}$$

and

$$\mathbb{M}_K[u] := \frac{1}{2\mathcal{L}^n(K)} \int_{\partial K} \left(u \otimes \nu_{\partial K} - \nu_{\partial K} \otimes u\right) d\mathcal{H}^{n-1}. \tag{3.8}$$

For BV maps we express the quantity $\mathbb{M}_K$ in (3.8) in terms of the skew-symmetric part of the total variation measure.

LEMMA 3.5. *Let A be a bounded open set with Lipschitz boundary. Then, for all $u \in BV(A; \mathbb{R}^n)$*

$$\int_{\partial A} \left(u \otimes \nu_{\partial A} - \nu_{\partial A} \otimes u\right) d\mathcal{H}^{n-1} = Du(A) - (Du)^t(A).$$

In particular, for all bounded, open, convex sets K, and for all $u \in BV(K; \mathbb{R}^n)$ we have

$$\mathbb{M}_K[u] = \frac{1}{2\mathcal{L}^n(K)} \left(Du(K) - (Du)^t(K)\right).$$

Proof. To prove the first equality we use the divergence theorem for scalar BV functions [3, Corollary 3.89]

$$\int_A v \operatorname{div} \Phi \, dx + \int_A \Phi \cdot dDv = \int_{\partial A} v \, \Phi \cdot \nu_{\partial A} \, d\mathcal{H}^{n-1},$$

for all $v \in BV(A)$, $\Phi \in \mathcal{C}^1(\bar{A}; \mathbb{R}^n)$, to get for any fixed $y \in \mathbb{R}^n$

$$\int_{\partial A} \left(u \otimes \nu_{\partial A}\right) y \, d\mathcal{H}^{n-1} = \int_{\partial A} (\nu_{\partial A} \cdot y) u \, d\mathcal{H}^{n-1} = Du(A)y,$$

and moreover

$$\int_{\partial A} \left(\nu_{\partial A} \otimes u\right) y \, d\mathcal{H}^{n-1} = \int_{\partial A} (u \cdot y) \nu_{\partial A} \, d\mathcal{H}^{n-1} = (Du)^t(A)y.$$

The very definition of $\mathbb{M}_K[u]$ in (3.8) and the previous computation provide the conclusion. $\qquad\square$

PROPOSITION 3.6. $\mathcal{R}$ *is an invariant set for* $\mathfrak{R}_K$.

Proof. If u is affine, i.e., $u = \mathbb{L}x + v$, $\mathbb{L} \in \mathbb{M}^{n \times n}_{skew}$, and $v \in \mathbb{R}^n$, we have $b_K[u] = v$ by a simple computation. Furthermore, by taking into account that $Du = \mathbb{L}$, Lemma 3.5 implies

$$\mathbb{M}_K[u] = \frac{1}{2}(\mathbb{L} - \mathbb{L}^t) = \mathbb{L}. \qquad \square$$

As the trace theorem implies the continuity of $\mathfrak{R}_K : BD(K) \to \mathcal{R}$, thanks to the previous result, Theorem 3.4 yields the existence of a constant $c > 0$ depending only on K such that for every $u \in BD(K)$

$$(3.9) \qquad \|u - \mathfrak{R}_K[u]\|_{L^{\frac{n}{n-1}}(K;\mathbb{R}^n)} \leq c\,|Eu|(K).$$

Remark 3.7. Let $u \in BD(\varepsilon K)$, where $\varepsilon > 0$ and K is a bounded, open, convex set K containing the origin. Then, for a constant c depending only on K we have

$$(3.10) \qquad \|u - \mathfrak{R}_{\varepsilon K}[u]\|_{L^1(\varepsilon K;\mathbb{R}^n)} \leq c\,\varepsilon|Eu|(\varepsilon K).$$

This follows from the Hölder inequality, (3.9), and a scaling argument by considering $u_\varepsilon(y) := u(\varepsilon y)$, $y \in K$, and noting that $\mathfrak{R}_{\varepsilon K}[u](\varepsilon y) = \mathfrak{R}_K[u_\varepsilon](y)$, $\|u - \mathfrak{R}_{\varepsilon K}[u]\|_{L^1(\varepsilon K;\mathbb{R}^n)} = \varepsilon^n\|u_\varepsilon - \mathfrak{R}_K[u_\varepsilon]\|_{L^1(K;\mathbb{R}^n)}$, and $|Eu|(\varepsilon K) = \varepsilon^{n-1}|Eu_\varepsilon|(K)$.

3.3. On the Cantor part of the symmetrized distributional derivative. Recently, the fine properties of BD functions have been complemented with the analogue of Alberti's rank-one theorem in the BV setting. More precisely, we recall the fundamental contribution by De Philippis and Rindler (cf. [21]).

THEOREM 3.8. *Let* $u \in BD(\Omega)$. *Then, for* $|E^c u|$-*a.e.* $x \in \Omega$

$$(3.11) \qquad \frac{\mathrm{d}Eu}{\mathrm{d}|Eu|}(x) = \frac{\eta(x) \odot \xi(x)}{|\eta(x) \odot \xi(x)|}$$

for some $\xi, \eta : \Omega \to \mathbb{S}^{n-1}$ *Borel vector fields.*

Next, we state a rigidity result for BD maps with constant polar vector that follows from that established in [20, Theorem 2.10(i)–(ii)] by taking into account that the measure on the right-hand side below is in addition positive (see also [20, Theorem 3.2]).

PROPOSITION 3.9. *If* $w \in BD_{loc}(\mathbb{R}^n)$ *is such that for some* $\eta, \xi \in \mathbb{R}^n$

$$(3.12) \qquad Ew = \frac{\eta \odot \xi}{|\eta \odot \xi|}|Ew|,$$

then

  (i) *if* $\eta \neq \pm\xi$

$$w(y) = \alpha_1(y \cdot \xi)\eta + \alpha_2(y \cdot \eta)\xi + \mathbb{L}y + v$$

    for some $\alpha_1, \alpha_2 \in BV_{loc}(\mathbb{R})$, $\mathbb{L} \in \mathbb{M}^{n \times n}_{skew}$, $v \in \mathbb{R}^n$;

  (ii) *if* $\eta = \pm\xi$

$$w(y) = \alpha(y \cdot \xi)\xi + \mathbb{L}y + v$$

    for some $\alpha \in BV_{loc}(\mathbb{R})$, $\mathbb{L} \in \mathbb{M}^{n \times n}_{skew}$, $v \in \mathbb{R}^n$.

The next lemma will be particularly useful when dealing with the antisymmetric part of the gradient in the Cantor part of the measure Eu (see (3.7) and (3.8) for the definitions of b_K and $\mathbb{M}_K$, respectively).

LEMMA 3.10. *Let K be a bounded, open, convex set containing the origin. For any $u \in BD(\Omega)$ and for $|E^c u|$-a.e. $x_0 \in \Omega$*

$$\lim_{\varepsilon \to 0} b_{K(x_0,\varepsilon)}[u] = u(x_0), \qquad \lim_{\varepsilon \to 0} \varepsilon |\mathbb{M}_{K(x_0,\varepsilon)}[u]| = 0.$$

Proof. Let $u \in BD(\Omega)$ be fixed. As noticed in the preliminaries $|E^c u|$-a.e. $x_0 \in \Omega$ is a point of approximate continuity for u, thus $b_{K(x_0,\varepsilon)}[u] \to u(x_0)$ as $\varepsilon \downarrow 0$.

For the second part of the statement, we use the computation in [1, Theorem 6.5, Corollary 6.7] implying that for $|E^c u|$-a.e. $x_0 \in \Omega$

$$(3.13) \qquad \lim_{\varepsilon \to 0} \fint_{B(x_0,\varepsilon)} |u - d_{B(x_0,\varepsilon)}[u]| \, \mathrm{d}y = 0,$$

where

$$d_{B(x_0,\varepsilon)}[u] := \fint_{\partial B(x_0,\varepsilon)} u(x) \, \mathrm{d}\mathcal{H}^{n-1}(x).$$

Let $x_0 \in C_u$ be a point for which (3.13) holds, and recall then that $\varepsilon^{1-n}|Eu|(B(x_0,\varepsilon)) \to 0$ as $\varepsilon \downarrow 0$. Define for any $v \in BD(\Omega)$ and for ε sufficiently small

$$\mathfrak{R}_\varepsilon^*[v](y) := d_{B(x_0,\varepsilon)}[v] + \mathbb{M}_{K(x_0,\varepsilon)}[v](y - x_0).$$

Arguing as in the proof of Proposition 3.6 it is immediate to see that $\mathcal{R}$ is an invariant set for $\mathfrak{R}_\varepsilon^*$. In particular, thanks to Theorem 3.4 we infer for all $v \in BD(\Omega)$ that

$$\|v - \mathfrak{R}_\varepsilon^*[v]\|_{L^1(B(x_0,\varepsilon);\mathbb{R}^n)} \leq c\,\varepsilon |Ev|(B(x_0,\varepsilon)),$$

where the constant c is independent from ε (this is obtained with a scaling argument similar to that in Remark 3.7). In particular, it follows that

$$\int_{B(x_0,\varepsilon)} |u(y) - d_{B(x_0,\varepsilon)}[u] - \mathbb{M}_{K(x_0,\varepsilon)}[u](y - x_0)| \, \mathrm{d}y \leq c\,\varepsilon |Eu|(B(x_0,\varepsilon)).$$

Therefore, by the triangular inequality we have

$$\fint_{B(x_0,\varepsilon)} |\mathbb{M}_{K(x_0,\varepsilon)}[u](y - x_0)| \, \mathrm{d}y \leq \fint_{B(x_0,\varepsilon)} |u(y) - d_{B(x_0,\varepsilon)}[u]| \, \mathrm{d}y + C\frac{|Eu|(B(x_0,\varepsilon))}{\varepsilon^{n-1}},$$

and thus by the choice of x_0 it follows that

$$\varepsilon \fint_{B(\underline{0},1)} |\mathbb{M}_{K(x_0,\varepsilon)}[u]z| \, \mathrm{d}z = \fint_{B(x_0,\varepsilon)} |\mathbb{M}_{K(x_0,\varepsilon)}[u](y - x_0)| \, \mathrm{d}y \to 0.$$

Notice that the quantity $\mathbb{M} \mapsto \fint_{B(\underline{0},1)} |\mathbb{M}z| \, \mathrm{d}z$ defines a norm on $\mathbb{M}^{n \times n}$, and thus for some constant C depending only on the dimension, we have

$$\varepsilon |\mathbb{M}_{K(x_0,\varepsilon)}[u]| \leq C\varepsilon \fint_{B(\underline{0},1)} |\mathbb{M}_{K(x_0,\varepsilon)}[u]z| \, \mathrm{d}z \to 0. \qquad \square$$

3.4. Change-of-base formulas. It is well-known that the chain rule formula does not hold in general for BD maps. We provide a simple variation of it that will be useful throughout the paper.

LEMMA 3.11. *Let $\mathbb{B} \in \mathbb{M}^{n \times n}$ be invertible, let $w \in BD(\Omega)$, and set*

$$\tilde{w}(y) := \mathbb{B}w(\mathbb{B}^t y).$$

Then, $\tilde{w} \in BD(\mathbb{B}^{-t}\Omega)$ and

$$E\tilde{w} = |\det \mathbb{B}|^{-1}\mathbb{B}\left(\mathbb{B}_\#^{-t}Ew\right)\mathbb{B}^t.$$

Moreover, if K is an open convex set and $v \in BD(K)$ we have

(3.14) $$\mathfrak{R}_{\mathbb{B}^{-t}K}[\tilde{v}](y) = \mathbb{B}\mathfrak{R}_K[v](\mathbb{B}^t y) \quad \text{for all } y \in \mathbb{R}^n.$$

Proof. Let $\varphi \in \mathcal{C}^\infty(\Omega; \mathbb{R}^n) \cap BD(\Omega)$ and define $\tilde{\varphi}(y) := \mathbb{B}\varphi(\mathbb{B}^t y)$. Clearly, $\tilde{\varphi} \in \mathcal{C}^\infty(\mathbb{B}^{-t}\Omega; \mathbb{R}^n)$, with $\nabla\tilde{\varphi}(y) = \mathbb{B}\nabla\varphi(\mathbb{B}^t y)\mathbb{B}^t$ and $e(\tilde{\varphi})(y) = \mathbb{B}e(\varphi)(\mathbb{B}^t y)\mathbb{B}^t$. Hence, the symmetrized distributional derivative of $\tilde{\varphi}$ is given by

$$E\tilde{\varphi} = |\det \mathbb{B}|^{-1}\mathbb{B}\left(\mathbb{B}_\#^{-t}(e(\varphi)\mathcal{L}^n \llcorner \Omega)\right)\mathbb{B}^t.$$

Finally, if $w \in BD(\Omega)$ we conclude by approximation of w by smooth maps in the BD strict topology.

The last assertion follows from a direct computation. $\qquad\square$

Remark 3.12. We shall often use Lemma 3.11 to reduce ourselves to the case in which the two vectors ξ, η in the polar decomposition of (3.11) are actually given by e_1 and e_2. To this aim the following remarks are useful. Let $w \in BD(K)$ be given by

$$w(y) := \psi_1(y \cdot \eta)\xi + \psi_2(y \cdot \xi)\eta$$

for some $\psi_1, \psi_2 \in BV(\mathbb{R})$ and for $\xi, \eta \in \mathbb{R}^n$ nonparallel unit vectors, i.e., $\eta \neq \pm\xi$. Consider any invertible matrix $\mathbb{B} \in \mathbb{M}^{n \times n}$ such that $\mathbb{B}\eta = e_1$, $\mathbb{B}\xi = e_2$, and the associated function

$$\tilde{w}(y) := \mathbb{B}\left(\psi_1\left(\mathbb{B}^t y \cdot \eta\right)\xi + \psi_2\left(\mathbb{B}^t y \cdot \xi\right)\eta\right) = \psi_1(y \cdot e_1)e_2 + \psi_2(y \cdot e_2)e_1.$$

Then, $\tilde{w} \in BD(\mathbb{B}^{-t}K)$ with $E\tilde{w} = |\det \mathbb{B}|^{-1}\mathbb{B}(\mathbb{B}_\#^{-t}Ew)\mathbb{B}^t$. Furthermore, since

$$\mathbb{B}_\#^{-t}Ew = \frac{\eta \odot \xi}{|\eta \odot \xi|}\mathbb{B}_\#^{-t}|Ew|$$

we then have

$$E\tilde{w} = |\det \mathbb{B}|^{-1}\frac{e_1 \odot e_2}{|\eta \odot \xi|}\mathbb{B}_\#^{-t}|Ew|,$$

in turn implying both

(3.15) $$E\tilde{w} = \frac{e_1 \odot e_2}{|e_1 \odot e_2|}|E\tilde{w}|$$

and

(3.16) $$|E\tilde{w}| = |\det \mathbb{B}|^{-1}\frac{|e_1 \odot e_2|}{|\eta \odot \xi|}\mathbb{B}_\#^{-t}|Ew|.$$

In particular, we conclude that

$$|E\tilde{w}|(\mathbb{B}^{-t}K) = |\det \mathbb{B}|^{-1}\frac{|e_1 \odot e_2|}{|\eta \odot \xi|}|Ew|(K).$$

3.5. On the cell problem defining $\mathfrak{m}$. The next two results clarify the link between $\mathfrak{m}$ and $\mathcal{F}$. They have been originally proved in [7] in the BV setting and then straightforwardly adapted to the BD setting in [24].

LEMMA 3.13 (Lemma 3.5 and Remark 3.6 in [7], Lemma 3.2 in [24]). *Let* $u \in BD(\Omega)$, *and set* $\mu := \mathcal{L}^n + |E^s u|$. *Then, for any bounded, open, convex set K containing the origin we have*

$$\lim_{r \to 0} \frac{\mathcal{F}(u, K(x_0, r))}{\mu(K(x_0, r))} = \lim_{r \to 0} \frac{\mathfrak{m}(u, K(x_0, r))}{\mu(K(x_0, r))} \qquad for \ \mu\text{-}a.e. \ x_0 \in \Omega.$$

LEMMA 3.14. *There exists a constant $C > 0$ such that for any $u_1, u_2 \in BD(\Omega)$, $A \in \mathcal{O}_\infty(\Omega)$*

$$|\mathfrak{m}(u_1; A) - \mathfrak{m}(u_2; A)| \leq C \int_{\partial A} |u_1 - u_2| \, \mathrm{d}\mathcal{H}^{n-1}.$$

Finally, we refine Lemma 3.14 as a consequence of assumptions (H4) and (H5). Following [7, Remark 3.10], for every K bounded, open, convex set containing the origin, for every $(\mathbb{A}, \mathrm{v}_0, \mathrm{v}, x_0, x) \in \mathbb{M}^{n \times n} \times (\mathbb{R}^n)^2 \times (\Omega)^2$, and for every $\varepsilon > 0$ small enough, hypothesis (H4) implies

$$|\mathfrak{m}(\mathrm{v} + \mathrm{v}_0 + \mathbb{A}(\cdot - x - x_0), K(x + x_0, \varepsilon)) - \mathfrak{m}(\mathrm{v}_0 + \mathbb{A}(\cdot - x_0), K(x_0, \varepsilon))|$$

$$(3.17) \qquad \leq C_K \Psi(|x| + |\mathrm{v}|) \left(1 + \frac{|\mathbb{A} + \mathbb{A}^t|}{2}\right) \varepsilon^n,$$

and, in turn, hypothesis (H5) implies

$$\left| \mathfrak{m}(\mathrm{v}_0 + \mathbb{A}(\cdot - x_0), K(x_0, \varepsilon)) - \mathfrak{m}\left(\mathrm{v}_0 + \frac{\mathbb{A} + \mathbb{A}^t}{2}(\cdot - x_0), K(x_0, \varepsilon)\right) \right|$$

$$(3.18) \qquad \leq C_K \Psi\left(\varepsilon \operatorname{diam}(K) \frac{|\mathbb{A} - \mathbb{A}^t|}{2}\right) \left(1 + \frac{|\mathbb{A} + \mathbb{A}^t|}{2}\right) \varepsilon^n$$

for some constant $C_K > 0$ depending on K only.

4. Analysis of the blow-ups of the Cantor part. In this section we show how to select a suitable blow-up limit at Cantor type points. To this aim we fix some notation: letting $u \in BD(\Omega)$, we may assume u to be extended to a map in $BD(\mathbb{R}^n)$ being Ω Lipschitz (cf. [23, Corollary 1.6.4]). By a slight abuse of notation we denote by u the extended function.

Given a bounded, open, convex set K containing the origin, for every $\varepsilon > 0$ and $x \in \Omega$ set $K(x, \varepsilon) := x + \varepsilon K$. For ε sufficiently small, consider the associated rescaled functions $u_{K,x,\varepsilon} : K \to \mathbb{R}^n$ given by

$$(4.1) \qquad u_{K,x,\varepsilon}(y) := \frac{u(x + \varepsilon y) - \mathfrak{R}_K[u(x + \varepsilon \cdot)](y)}{\varepsilon \frac{|Eu|(K(x,\varepsilon))}{\mathcal{L}^n(K(x,\varepsilon))}},$$

where $\mathfrak{R}_K$ is defined in (3.6). Clearly, we may assume that $|Eu|(K(x, \varepsilon)) > 0$ for all $\varepsilon \in (0, \operatorname{dist}(x, \partial\Omega))$; otherwise u would be an infinitesimal rigid motion around x (cf. (3.5) in Theorem 3.4). We first analyze basic compactness properties of the rescaled family $\{u_{K,x,\varepsilon}\}_{\varepsilon > 0}$.

PROPOSITION 4.1. *For $|Eu|$-a.e. $x \in \Omega$ there exist a sequence $\varepsilon_i \downarrow 0$, a map $w \in BD(K)$, and a measure $\gamma \in \mathrm{Tan}(Eu, x)$ such that*

(i) $\{u_{K,x,\varepsilon_i}\}_{i \in \mathbb{N}}$ *converges to w strictly in $BD(K)$, $\mathfrak{R}_K[w] = \underline{0}$;*

(ii) $\{\frac{1}{\mathcal{L}^n(K)} Eu_{K,x,\varepsilon_i}\}_{i \in \mathbb{N}}$ *converges to γ weakly* in $\mathcal{M}(\bar{K}; \mathbb{R}^{n \times n})$, $|\gamma|(K) = |\gamma(K)|$ $= 1$, $|\gamma|(\partial K) = 0$, $|\gamma| \in \mathrm{Tan}(|Eu|, x)$, $\gamma = \frac{dEu}{d|Eu|}(x)|\gamma|$ $|\gamma|$-a.e. in $\mathbb{R}^n$, and $Ew = \mathcal{L}^n(K)\gamma \llcorner K$.*

Proof. We prove the conclusion for all Lebesgue points of the polar vector $\Omega \ni x \mapsto \frac{dEu}{d|Eu|}(x)$. A simple computation shows that

$$(4.2) \qquad Eu_{K,x,\varepsilon} = \mathcal{L}^n(K) \frac{F_\#^{x,\varepsilon} Eu}{F_\#^{x,\varepsilon}|Eu|(K)}$$

(recall the definition of $F_\#^{x,\varepsilon}$ in (3.1)) so that $|Eu_{K,x,\varepsilon}|(K) = \mathcal{L}^n(K)$. Lemma 3.2 yields for a sequence $\varepsilon_i \downarrow 0$ that $\{\frac{1}{\mathcal{L}^n(K)} Eu_{K,x,\varepsilon_i}\}_{i \in \mathbb{N}}$ converges weakly* in $\mathcal{M}(\bar{K}; M^{n \times n})$ to some $\gamma \in \mathrm{Tan}(Eu, x)$, with $|\gamma|(K) = 1$, $|\gamma|(\partial K) = 0$, $|\gamma| \in \mathrm{Tan}(|Eu|, x)$ and such that $\gamma = \frac{dEu}{d|Eu|}(x)|\gamma|$ $|\gamma|$-a.e. in $\mathbb{R}^n$ (for the last two claims see [3, Theorem 2.44], [43, Lemma 10.4]), then $|\gamma(K)| = 1$.

Moreover, by taking into account Remark 3.7 we get

$$\|u_{K,x,\varepsilon}\|_{L^1(K;\mathbb{R}^n)} = \frac{\mathcal{L}^n(K)}{\varepsilon |Eu|(K(x,\varepsilon))} \|u(x + \cdot) - \mathfrak{R}_{\varepsilon K}[u(x + \cdot)](\cdot)\|_{L^1(\varepsilon K;\mathbb{R}^n)},$$

so that for some constant depending only on K

$$\|u_{K,x,\varepsilon}\|_{L^1(K;\mathbb{R}^n)} \leq c\,\mathcal{L}^n(K).$$

The compact embedding $BD(K) \hookrightarrow L^1(K;\mathbb{R}^n)$ yields that we can extract a subsequence (not relabeled) with u_{K,x,ε_i} converging in $L^1(K;\mathbb{R}^n)$ to some limit map w belonging to $BD(K)$ satisfying $\mathfrak{R}_K[w] = \underline{0}$. Therefore, Eu_{K,x,ε_i} converge to Ew weakly* in $\mathcal{M}(K;\mathbb{R}^{n \times n})$, and thus $Ew = \mathcal{L}^n(K)\gamma \llcorner K$. The strict convergence in $BD(K)$ of $\{u_{K,x,\varepsilon_i}\}_{i \in \mathbb{N}}$ to w then follows at once. $\qquad \square$

In what follows, any map w given by Proposition 4.1 will be termed a blow-up limit. If x is a point of approximate differentiability or a jump point, usually introduced with a different definition of the rescaled maps, blow-up limits are well-known to be unique. In turn, this implies that the Radon–Nikodým derivative of the functional $\mathcal{F}$ with respect to $|Eu|$ in such points can be straightforwardly characterized in terms of asymptotic Dirichlet problems with boundary values given by the blow-up limit itself (cf. Lemma 3.13). In contrast, if $x \in C_u$ is a point satisfying (3.11), usually referred to as a Cantor type point, blow-up limits are in general not unique. In order to overcome this difficulty, we use assumption (H4) and moreover we perform a double blow up procedure. Indeed, by means of the latter argument, we can reduce ourselves to the case of a two-dimensional BV map which is affine in one direction.

The strategy of the proof is a slight variation of [22, Lemma 2.14], which is originally worked out in the context of generalized Young measures. We basically follow the lines of such proof, though not using Young measures, by incorporating also the need of selecting a sequence preserving the mass along the blow-up process. We shall improve upon the structure of blow-ups in Proposition 4.6 in section 4.2.

4.1. A double blow-up procedure. We introduce some notation necessary for the blow-up procedure. Given a couple of vectors $\xi, \eta \in \mathbb{S}^{n-1}$ (possibly $\xi = \pm\eta$), consider an orthonormal basis ζ_i of $\mathrm{span}\{\xi, \eta\}^\perp$ (thus for $n = 2$ either $\mathrm{span}\{\xi, \eta\}^\perp = \{\underline{0}\}$ if $\xi \neq \pm\eta$ or $i = 1$ if $\xi = \pm\eta$, and for $n \geq 3$ either $1 \leq i \leq n - 2$ if $\xi \neq \pm\eta$ or

$1 \leq i \leq n-1$ if $\xi = \pm\eta$), then for all $\rho > 0$ define the bounded, open, convex set containing the origin

$$(4.3) \qquad P_\rho^{\xi,\eta} := \left\{ y \in \mathbb{R}^n : |y \cdot \eta| \leq \rho/2, \; |y \cdot \xi| \leq 1/2, \; |y \cdot \zeta_i| \leq 1/2 \text{ for } 1 \leq i \leq n-2 \right\}$$

if $\xi \neq \pm\eta$, and otherwise if $\xi = \pm\eta$

$$(4.4) \qquad P_\rho^{\xi,\eta} := \left\{ y \in \mathbb{R}^n : |y \cdot \eta| \leq \rho/2, \; |y \cdot \zeta_i| \leq 1/2 \text{ for } 1 \leq i \leq n-1 \right\}.$$

We underline that the role of η and ξ is not symmetric in the definition of $P_\rho^{\xi,\eta}$ (in this respect see the comments right before Case 1 in the ensuing proof). Moreover, we do not highlight the dependence of $P_\rho^{\xi,\eta}$ on ζ_i so as not to further overburden the notation. In any case the specific choice of ζ_i is not relevant for the arguments that follow.

With this notation at hand, we can state the key result to prove the integral representation of the Cantor part (recall the definitions of $F_\#^{x,\varepsilon} Eu$ given in (3.1) and that of $u_{K,x,\varepsilon}$ given in (4.1)).

PROPOSITION 4.2 (blow-up at $|E^c u|$-a.e. point). *Let $u \in BD(\Omega)$. Then for $|E^c u|-$ a.e. $x \in \Omega$ and for every $\rho > 0$ there exist an infinitesimal sequence $\{\varepsilon_i\}_{i\in\mathbb{N}}$, vectors $\xi, \eta \in \mathbb{S}^{n-1}$, a bounded, open, convex set containing the origin $P_\rho^x := P_\rho^{\xi,\eta}$ (cf. (4.3) and (4.4)), and a map $v_\rho \in BV(P_\rho^x; \mathbb{R}^n)$ such that $u_{P_\rho^x, x, \varepsilon_i}$ converge to v_ρ strict in $BD(P_\rho^x)$, where*

$$(4.5) \qquad v_\rho(y) = \bar{\psi}_\rho(y \cdot \eta)\xi + (y \cdot \xi)\bar{\beta}_\rho\, \eta + \mathbb{L}_\rho y + \mathrm{v}_\rho,$$

for some $\bar{\psi}_\rho \in BV_{loc}(\mathbb{R})$, $\bar{\beta}_\rho \in \mathbb{R}$, $\mathrm{v}_\rho \in \mathbb{R}^n$, and $\mathbb{L}_\rho \in \mathbb{M}_{skew}^{n \times n}$. Moreover, v_ρ satisfies

$$(4.6) \qquad \mathfrak{R}_{P_\rho^x}[v_\rho] = \underline{0}, \qquad |Ev_\rho|(P_\rho^x) = \mathcal{L}^n(P_\rho^x),$$

and

$$\frac{\mathrm{d}Ev_\rho}{\mathrm{d}|Ev_\rho|}(y) = \frac{\eta \odot \xi}{|\eta \odot \xi|}, \qquad |Ev_\rho|\text{-a.e. on } P_\rho^x.$$

Proof. We divide the proof into two steps, each corresponding to a blow-up procedure.

First blow-up. We perform a first blow-up in a point x satisfying all the conditions listed in (I)–(III) that follow. More precisely, consider the subset of points $x \in C_u$ (so that x is a point of approximate continuity of u (see (3.3))) having the following additional properties:

(I) $\frac{\mathrm{d}Eu}{\mathrm{d}|Eu|}(x) = \frac{\mathrm{d}E^c u}{\mathrm{d}|E^c u|}(x) = \frac{\eta(x) \odot \xi(x)}{|\eta(x) \odot \xi(x)|}$, for some vectors $\xi(x), \eta(x) \in \mathbb{S}^{n-1}$, and $\mathrm{Tan}(|Eu|, x) = \mathrm{Tan}(|E^c u|, x)$.

(II) Proposition 4.1 holds true with $K := \mathbb{B}^t Q(\underline{0}, \rho)$ if $\eta(x) \neq \pm\xi(x)$, where $\mathbb{B}$ is any invertible matrix such that $\mathbb{B}\eta(x) = \mathrm{e}_1$ and $\mathbb{B}\xi(x) = \mathrm{e}_2$, and $K := P_\rho^{\xi(x),\eta(x)}$ if $\eta(x) = \pm\xi(x)$: we extract a subsequence (not relabeled) such that u_{K,x,ε_i} converge to some map w strict in $BD(K)$, with $\mathfrak{R}_K[w] = \underline{0}$, $\{\frac{1}{\mathcal{L}^n(K)}Eu_{K,x,\varepsilon_i}\}_{i\in\mathbb{N}}$ converges weakly* in $\mathcal{M}(\bar{K}; \mathbb{R}^{n\times n})$ to some $\gamma \in \mathrm{Tan}(Eu, x)$ such that $|\gamma|(K) = |\gamma|(K) = 1$, $|\gamma|(\partial K) = 0$, $|\gamma| \in \mathrm{Tan}(|Eu|, x)$, $\gamma = \frac{dEu}{d|Eu|}(x)|\gamma|$ $|\gamma|$-a.e. in $\mathbb{R}^n$, and $Ew = \mathcal{L}^n(K)\gamma \llcorner K$.

(III) $\mathrm{Tan}(|Ew|, z) \subseteq \mathrm{Tan}(|E^c u|, x)$ for all $z \in \mathrm{spt}\,|Ew|$.

Notice that the set of points where either (I) or (II) or (III) fails is $|E^c u|$-negligible thanks to Theorem 3.8, to the locality of Preiss' tangent space to a measure, to Proposition 4.1 itself, and to Theorem 3.3. Proposition 3.9 describes the structure of the blow-up limit w in (II) in detail:

- if $\eta(x) \neq \pm\xi(x)$: we can find two maps $\alpha_1, \alpha_2 \in BV_{loc}(\mathbb{R})$, $\mathbb{L}_\rho \in \mathbb{M}_{skew}^{n \times n}$, and $v_\rho \in \mathbb{R}^n$, such that

$$w(y) = \alpha_1(y \cdot \eta(x))\xi(x) + \alpha_2(y \cdot \xi(x))\eta(x) + \mathbb{L}_\rho y + v_\rho \, ;$$

- if $\eta(x) = \pm\xi(x)$: we can find a map $\alpha \in BV_{loc}(\mathbb{R})$, $\mathbb{L}_\rho \in \mathbb{M}_{skew}^{n \times n}$, and $v_\rho \in \mathbb{R}^n$, such that

$$w(y) = \alpha(y \cdot \xi(x))\xi(x) + \mathbb{L}_\rho y + v_\rho \, .$$

Thus, if $\eta(x) = \pm\xi(x)$ we conclude by setting $v_\rho := w$ and with $P_\rho^x := P_\rho^{\xi(x),\eta(x)}$. Otherwise, if $\eta(x) \neq \pm\xi(x)$, we are forced to take a second blow-up to prove that (at least) one between the α_k's can be taken affine.

Second blow-up if $\boldsymbol{\eta(x) \neq \pm\xi(x)}$. First, we change variables by means of the invertible matrix $\mathbb{B}$ introduced in item (II) above. Following Remark 3.12, with $\tilde{\mathbb{L}}_\rho := \mathbb{B}\mathbb{L}_\rho\mathbb{B}^t$ and $\tilde{v}_\rho := \mathbb{B}v_\rho$, we consider the associated map

$$\tilde{w}(y) = \alpha_1(y \cdot e_1)e_2 + \alpha_2(y \cdot e_2)e_1 + \tilde{\mathbb{L}}_\rho y + \tilde{v}_\rho \, .$$

We blow up $\tilde{w}$ around a suitable point $y \in \mathbb{B}^{-t}K = Q(\underline{0}, \rho)$, distinguishing two cases depending on the distributional derivatives of the α_k's. We note that the vectors η and ξ in the statement correspond exactly to the two vectors provided by the polar decomposition of $E^c u$ at x, $\eta(x)$, and $\xi(x)$, respectively, if $D^s\alpha_1 \neq 0$. In this case we also set $P_\rho^x := P_\rho^{\xi(x),\eta(x)}$, the latter set being introduced in (4.3). Instead, if $D^s\alpha_1 = 0$ and $D^s\alpha_2 \neq 0$, then η corresponds to $\xi(x)$ and ξ to $\eta(x)$, and $P_\rho^x := P_\rho^{\eta(x),\xi(x)}$. Finally, if both $D^s\alpha_1 = D^s\alpha_2 = 0$, we choose $\eta = \eta(x)$ and $\xi = \xi(x)$ and set $P_\rho^x := P_\rho^{\xi(x),\eta(x)}$ (actually, the opposite choice would be fine as well).

Case 1: Either $D^s\alpha_1 \neq 0$ or $D^s\alpha_2 \neq 0$. Without loss of generality we may assume that $D^s\alpha_1 \neq 0$. Set $\tilde{P}_\rho^x := \mathbb{B}^{-t}P_\rho^x$ (the matrix $\mathbb{B}$ has been introduced above), and select a point $y \in Q(\underline{0}, \rho)$ such that

(a) $\frac{dE\tilde{w}}{d|E\tilde{w}|}(y) = \frac{e_1 \odot e_2}{|e_1 \odot e_2|}$, and $\delta^{-n}|E\tilde{w}|(\tilde{P}_\rho^x(y, \delta)) \to +\infty$ as $\delta \downarrow 0$;

(b) Proposition 4.1 holds true: $\tilde{w}_{\tilde{P}_\rho^x, y, \delta_i}$, for some sequence $\delta_i \downarrow 0$, converge to a map $\tilde{g}_\rho$ strict in $BD(\tilde{P}_\rho^x)$ with $\mathfrak{R}_{\tilde{P}_\rho^x}[\tilde{g}_\rho] = \underline{0}$, and $\{\frac{1}{\mathcal{L}^n(\tilde{P}_\rho^x)}E\tilde{w}_{\tilde{P}_\rho^x, y, \delta_i}\}_{i \in \mathbb{N}}$ converges weakly* in $\mathcal{M}(\bar{\tilde{P}}_\rho^x; \mathbb{R}^{n \times n})$ to some $\tilde{\gamma} \in \text{Tan}(E\tilde{w}, y)$ such that $|\tilde{\gamma}|(\tilde{P}_\rho^x) = |\tilde{\gamma}(\tilde{P}_\rho^x)| = 1$, $|\tilde{\gamma}|(\partial\tilde{P}_\rho^x) = 0$, $|\tilde{\gamma}| \in \text{Tan}(|E\tilde{w}|, y)$, and $\tilde{\gamma} = \frac{dE\tilde{w}}{d|E\tilde{w}|}(y)|\tilde{\gamma}|$ $|\tilde{\gamma}|$-a.e. on $\mathbb{R}^n$, $E\tilde{g}_\rho = \mathcal{L}^n(\tilde{P}_\rho^x)\tilde{\gamma} \llcorner \tilde{P}_\rho^x$;

(c) $\lim_{r\to 0} \fint_{y \cdot e_2 - r}^{y \cdot e_2 + r} |\alpha_2'(s) - \alpha_2'(y \cdot e_2)| \, ds = \lim_{r\to 0} \frac{1}{2r}|D^s\alpha_2|(y \cdot e_2 - r, y \cdot e_2 + r) = 0$.

Since for $\mathcal{L}^1$-a.e. $t \in \mathbb{R}$ it holds that

$$\lim_{r\to 0} \fint_{t-r}^{t+r} |\alpha_2'(s) - \alpha_2'(t)| \, ds = 0, \quad \lim_{r\to 0} \frac{1}{2r}|D^s\alpha_2|(t-r, t+r) = 0,$$

condition (c) is true for $|D^s\alpha_1| \otimes \mathcal{L}^{n-1}$-a.e. $y \in Q(\underline{0}, \rho)$. First, note that $|D^s\alpha_1| \otimes \mathcal{L}^{n-1}$ is nontrivial by assumption; then if $I \subseteq \mathbb{R}$ is the subset of points of full $\mathcal{L}^1$ measure for which the previous two conditions hold true, we conclude that

$$\left(|D^s\alpha_1| \otimes \mathcal{L}^{n-1}\right)\left((-\rho/2, \rho/2) \times ((-\rho/2, \rho/2) \setminus I) \times (-\rho/2, \rho/2)^{n-2}\right) = 0.$$

On the other hand, conditions (a)–(b) hold $|E^s\tilde{w}|$-a.e. on $Q(\underline{0}, \rho)$ in view of Theorem 3.8, the decomposition of Ew in (3.2), and Proposition 4.1 itself. Since the measures $|D^s\alpha_1| \otimes \mathcal{L}^{n-1}$ and $|D\alpha_1| \otimes |D^s\alpha_2| \otimes \mathcal{L}^{n-2}$ are mutually singular, the measure

$|D^s\alpha_1| \otimes \mathcal{L}^{n-1}$ is absolutely continuous with respect to $|E^s\tilde{w}|$, and thus we conclude that (a)–(c) hold for $|D^s\alpha_1| \otimes \mathcal{L}^{n-1}$-a.e. $y \in Q(\underline{0}, \rho)$.

Then, fixing a point $y \in Q(\underline{0}, \rho)$ for which (a)–(c) are satisfied, we show that the second blow-up limit $\tilde{g}_\rho$ satisfies

(4.7)
$$\tilde{g}_\rho(z) = \psi_\rho(z \cdot \mathrm{e}_1)\,\mathrm{e}_2 + \tilde{\mathbb{L}}_\rho z + \tilde{\mathrm{v}}_\rho$$

for some $\psi_\rho \in BV_{loc}(\mathbb{R})$, $\tilde{\mathrm{v}}_\rho \in \mathbb{R}^n$, and $\tilde{\mathbb{L}}_\rho \in \mathbb{M}^{n\times n}_{skew}$. We make a slight abuse of notation since the latter two quantities may be different from those in the definition of $\tilde{w}$, but as their role is inessential we keep the same symbols. To this aim, we split $E\tilde{w}_{\tilde{P}^x_\rho, y, \delta_i}$ as

$$E\tilde{w}_{\tilde{P}^x_\rho, y, \delta_i} = E\tilde{w}^{(1)}_i + E\tilde{w}^{(2)}_i\,,$$

where we have set

$$\tilde{w}^{(1)}_i(z) := \frac{\alpha_1((y + \delta_i z) \cdot \mathrm{e}_1)\mathrm{e}_2 - \mathfrak{R}_{\tilde{P}^x_\rho}[\alpha_1((y + \delta_i \cdot) \cdot \mathrm{e}_1)\mathrm{e}_2](z)}{\delta_i \frac{|E\tilde{w}|(\tilde{P}^x_\rho(y, \delta_i))}{\mathcal{L}^n(\tilde{P}^x_\rho(y, \delta_i))}}\,,$$

and

$$\tilde{w}^{(2)}_i(z) := \frac{\alpha_2((y + \delta_i z) \cdot \mathrm{e}_2)\mathrm{e}_1 - \mathfrak{R}_{\tilde{P}^x_\rho}[\alpha_2((y + \delta_i \cdot) \cdot \mathrm{e}_2)\mathrm{e}_1](z)}{\delta_i \frac{|E\tilde{w}|(\tilde{P}^x_\rho(y, \delta_i))}{\mathcal{L}^n(\tilde{P}^x_\rho(y, \delta_i))}}\,.$$

First, we claim that as $i \uparrow +\infty$

(4.8)
$$|E\tilde{w}^{(2)}_i|(\tilde{P}^x_\rho) \to 0\,.$$

Indeed, note that

$$E\tilde{w}^{(2)}_i = \mathcal{L}^n(\tilde{P}^x_\rho)\frac{\mathrm{e}_1 \odot \mathrm{e}_2}{|E\tilde{w}|(\tilde{P}^x_\rho(y, \delta_i))}\Big(\delta_i^n \alpha_2'\,((y + \delta_i \cdot) \cdot \mathrm{e}_2)\,\mathcal{L}^n \llcorner \tilde{P}^x_\rho$$
$$+ F^{y, \delta_i}_\#\big((\mathcal{L}^1 \otimes D^s\alpha_2 \otimes \mathcal{L}^{n-2}) \llcorner \tilde{P}^x_\rho\big)\Big)\,.$$

On setting $t_i := 1/|E\tilde{w}|(\tilde{P}^x_\rho(y, \delta_i))$, and noting that $t_i\delta_i^n$ is infinitesimal in view of the choice of y above, if $\varphi \in \mathcal{C}^0_c(\tilde{P}^x_\rho; \mathbb{R}^n)$ and $\alpha := \alpha_2'(y \cdot \mathrm{e}_2)$, we conclude that

$$t_i\delta_i^n\left|\int_{\tilde{P}^x_\rho}\varphi(z) : (\mathrm{e}_1 \odot \mathrm{e}_2)\alpha_2'((y + \delta_i z) \cdot \mathrm{e}_2)\,\mathrm{d}z\right| \le t_i\int_{\tilde{P}^x_\rho(y, \delta_i)}\left|\varphi\left(\frac{x - y}{\delta_i}\right)\right||\alpha_2'(x \cdot \mathrm{e}_2)|\,\mathrm{d}x$$

$$\le t_i\|\varphi\|_\infty\int_{\tilde{P}^x_\rho(y, \delta_i)}|\alpha_2'(x \cdot \mathrm{e}_2) - \alpha|\,\mathrm{d}x + t_i\delta_i^n|\alpha|\|\varphi\|_\infty\mathcal{L}^n(\tilde{P}^x_\rho)$$

$$= \rho\|\varphi\|_\infty t_i\delta_i^n\fint_{(y\cdot\mathrm{e}_2)-\delta_i/2}^{(y\cdot\mathrm{e}_2)+\delta_i/2}|\alpha_2'(s) - \alpha|\,\mathrm{d}s + o(1)\,,$$

which vanishes as $i \uparrow +\infty$ (cf. item (b) above). Arguing similarly, we get that

$$t_i\left|\int_{\tilde{P}^x_\rho}\varphi(z)\,\mathrm{d}F^{y, \delta_i}_\#\big((\mathcal{L}^1 \otimes D^s\alpha_2 \otimes \mathcal{L}^{n-2}) \llcorner \tilde{P}^x_\rho\big)(z)\right|$$

$$\le \rho\|\varphi\|_\infty\mathcal{L}^n(\tilde{P}^x_\rho)t_i\delta_i^{n-1}|D^s\alpha_2|(y \cdot \mathrm{e}_2 - \delta_i/2, y \cdot \mathrm{e}_2 + \delta_i/2)$$

is infinitesimal for $i \uparrow +\infty$, as well (cf. item (c) above).

In conclusion, (4.8) implies that $\tilde{w}_i^{(2)}$ converges to 0 strongly in $BD(\tilde{P}_\rho^x)$ in view of Remark 3.7, and then the limit of $\tilde{w}_{\tilde{P}_\rho^x, y, \delta_i}$ is equal to that of $\tilde{w}_i^{(1)}$. Moreover, the latter coincides with the $L^1(\tilde{P}_\rho^x; \mathbb{R}^n)$ limit of

$$\frac{\alpha_1((y + \delta_i z) \cdot e_1) e_2 - \mathfrak{R}_{\tilde{P}_\rho^x}[\alpha_1((y + \delta_i \cdot) \cdot e_1) e_2](z)}{\delta_i \frac{|E(\alpha_1(\cdot \cdot e_1) e_2)|(\tilde{P}_\rho^x(y, \delta_i))}{\mathcal{L}^n(\tilde{P}_\rho^x(y, \delta_i))}}$$

as $|E\tilde{w}_{\tilde{P}_\rho^x, y, \delta_i}|(\tilde{P}_\rho^x) = \mathcal{L}^n(\tilde{P}_\rho^x)$, and $|E\tilde{w}_i^{(2)}|(\tilde{P}_\rho^x)$ is infinitesimal. Thus, by the blow-up theory for BV functions (cf. [3, Proposition 3.77, Theorem 3.95]), Lemma 3.5, and Remark 3.7 we conclude (4.7) with

$$\tilde{g}_\rho(z) = \psi_\rho(z \cdot e_1) e_2 - \frac{1}{\sqrt{2}}(e_2 \otimes e_1 - e_1 \otimes e_2) z.$$

Case 2: $D^s \alpha_1 = D^s \alpha_2 = 0$. In this case $\tilde{w} \in W^{1,1}(\tilde{P}_\rho^x, \mathbb{R}^n)$, where $\tilde{P}_\rho^x := \mathbb{B}^{-t} P_\rho^x$ (the matrix $\mathbb{B}$ has been introduced in item (II) above). Arguing as in Case 1, we select a point $y \in Q(\underline{0}, \rho)$ such that
(a') $\frac{dE\tilde{w}}{d|E\tilde{w}|}(y) = \frac{e_1 \odot e_2}{|e_1 \odot e_2|}$;
(b') Proposition 4.1 holds true: $\tilde{w}_{\tilde{P}_\rho^x, y, \delta_i}$, for some sequence $\delta_i \downarrow 0$, converge to a map $\tilde{g}_\rho$ strict in $BD(\tilde{P}_\rho^x)$ with $\mathfrak{R}_{\tilde{P}_\rho^x}[\tilde{g}_\rho] = \underline{0}$, and $\{\frac{1}{\mathcal{L}^n(\tilde{P}_\rho^x)} E\tilde{w}_{\tilde{P}_\rho^x, y, \delta_i}\}_{i \in \mathbb{N}}$ converges weakly* in $\mathcal{M}(\bar{\tilde{P}}_\rho^x; \mathbb{R}^{n \times n})$ to some $\tilde{\gamma} \in \mathrm{Tan}(E\tilde{w}, y)$ such that $|\tilde{\gamma}|(\tilde{P}_\rho^x) = |\tilde{\gamma}(\tilde{P}_\rho^x)| = 1$, $|\tilde{\gamma}|(\partial \tilde{P}_\rho^x) = 0$, $|\tilde{\gamma}| \in \mathrm{Tan}(|E\tilde{w}|, y)$, $\tilde{\gamma} = \frac{dE\tilde{w}}{d|E\tilde{w}|}(y)|\tilde{\gamma}|$ $|\tilde{\gamma}|$-a.e. on $\mathbb{R}^n$, and $E\tilde{g}_\rho = \mathcal{L}^n(\tilde{P}_\rho^x) \tilde{\gamma} \llcorner \tilde{P}_\rho^x$;
(c') $\lim_{r \to 0} \fint_{(y \cdot e_k) - r}^{(y \cdot e_k) + r} |\alpha_k'(s) - \alpha_k'(y \cdot e_k)| \, ds = 0$ for $k \in \{1, 2\}$;
(d') $|\alpha_1'(y \cdot e_1) + \alpha_2'(y \cdot e_2)| \neq 0$.
Note that conditions (a')–(c') hold for $\mathcal{L}^n$-a.e. $y \in Q(\underline{0}, \rho)$, and (d') for a set of positive Lebesgue measure in $Q(\underline{0}, \rho)$ as $|E\tilde{w}|(Q(\underline{0}, \rho)) = \frac{|e_1 \odot e_2|}{|\eta \odot \xi|} \mathcal{L}^n(Q(\underline{0}, \rho)) \neq 0$ thanks to (3.16) and (II). As a consequence of all these conditions, $\delta^{-n}|E\tilde{w}|(\tilde{P}_\rho^x(y, \delta)) \to |e(\tilde{w})(y)| \mathcal{L}^n(\tilde{P}_\rho^x) \neq 0$ as $\delta \downarrow 0$. Thus, by blowing up the function $\tilde{w}$ at one such point along the sequence of radii δ_i given by (b'), we may infer that, up to extracting a subsequence not relabeled, $\tilde{w}_{\tilde{P}_\rho^x, y, \delta_i}$ converge strictly in $BD(\tilde{P}_\rho^x)$ to

$$\tilde{g}_\rho(z) = \frac{\alpha_1'(y \cdot e_1) + \alpha_2'(y \cdot e_2)}{2|e(\tilde{w})(y)|}\big((z \cdot e_1) e_2 + (z \cdot e_2) e_1\big).$$

To get the latter formula, we use that in this setting α_1 and α_2 are $W_{loc}^{1,1}(\mathbb{R})$ and that condition (c') and Lemma 3.5 hold true.

Conclusion in case $\eta(x) \neq \pm \xi(x)$. In both Case 1 and Case 2 we have selected a point $y \in Q(\underline{0}, \rho)$, a function $\tilde{g}_\rho \in BD(\tilde{P}_\rho^x)$, and a measure $\tilde{\gamma} \in \mathrm{Tan}(E\tilde{w}, y)$ satisfying that $\tilde{g}_\rho$ is affine in (at least) one direction between e_1 and e_2 (cf. (4.7)) and $|\tilde{\gamma}|(\tilde{P}_\rho^x) = |\tilde{\gamma}(\tilde{P}_\rho^x)| = 1$, $|\tilde{\gamma}|(\partial \tilde{P}_\rho^x) = 0$, $|\tilde{\gamma}| \in \mathrm{Tan}(|E\tilde{w}|, y)$, $\tilde{\gamma} = \frac{e_1 \odot e_2}{|e_1 \odot e_2|}|\tilde{\gamma}|$ $|\tilde{\gamma}|$-a.e. on $\mathbb{R}^n$, and $E\tilde{g}_\rho = \mathcal{L}^n(\tilde{P}_\rho^x) \tilde{\gamma} \llcorner \tilde{P}_\rho^x$.

Lemma 3.11 implies that $\mathrm{Tan}(Ew, \mathbb{B}^t y) = \mathbb{B}^{-1}\big(\mathbb{B}_\#^t \mathrm{Tan}(E\tilde{w}, y)\big)\mathbb{B}^{-t}$. Therefore, by setting $g_\rho := \mathbb{B}^{-1}\tilde{g}_\rho(\mathbb{B}^{-t} \cdot)$ and $\gamma := \mathbb{B}^{-1}\big(\mathbb{B}_\#^t \tilde{\gamma}\big)\mathbb{B}^{-t}$ we deduce that $\gamma \in \mathrm{Tan}(Ew, \mathbb{B}^t y)$

$\subseteq \mathrm{Tan}(E^c u, x)$ and that $|\gamma| \in \mathrm{Tan}(|Ew|, \mathbb{B}^t y) \subseteq \mathrm{Tan}(|E^c u|, x)$ in view of condition (III). Moreover, $Eg_\rho = \mathcal{L}^n(P_\rho^x) \gamma \llcorner P_\rho^x$.

Let $\{r_i\}_{i \in \mathbb{N}}$ be a sequence of radii and $\{c_i\}_{i \in \mathbb{N}}$ of positive constants such that $c_i F_\#^{x, r_i} Eu$ converge locally weakly* to γ in $\mathcal{M}_{loc}(\mathbb{R}^n; \mathbb{R}^{n \times n})$, and $c_i F_\#^{x, r_i} |Eu|$ converge locally weakly* to $|\gamma|$ on $\mathcal{M}(\mathbb{R}^n)$. Then, $|\gamma|(\partial P_\rho^x) = 0$, by taking into account that $|\tilde\gamma|(\partial \tilde P_\rho^x) = 0$. Thus, Lemma 3.1 yields

$$c_i F_\#^{x, r_i} |Eu|(P_\rho^x) \to |Eg_\rho|(P_\rho^x) \stackrel{(3.16)}{=} |\det \mathbb{B}| \frac{|\eta \odot \xi|}{|\mathrm{e}_1 \odot \mathrm{e}_2|} \mathcal{L}^n(\tilde P_\rho^x) = \frac{|\eta \odot \xi|}{|\mathrm{e}_1 \odot \mathrm{e}_2|} \mathcal{L}^n(P_\rho^x).$$

Hence, $\mathcal{L}^n(P_\rho^x) \frac{F_\#^{x, r_i} Eu}{F_\#^{x, r_i} |Eu|(P_\rho^x)} \stackrel{*}{\rightharpoonup} \frac{|\mathrm{e}_1 \odot \mathrm{e}_2|}{|\eta \odot \xi|} Eg_\rho$ in $\mathcal{M}(P_\rho^x; \mathbb{R}^{n \times n})$, and $\mathcal{L}^n(P_\rho^x) \frac{F_\#^{x, r_i} |Eu|}{F_\#^{x, r_i} |Eu|(P_\rho^x)}$ $\stackrel{*}{\rightharpoonup} \frac{|\mathrm{e}_1 \odot \mathrm{e}_2|}{|\eta \odot \xi|} |Eg_\rho|$ in $\mathcal{M}(P_\rho^x)$. Furthermore, note that the rescaled maps $u_{P_\rho^x, x, r_i}$ converge strongly in $L^1(P_\rho^x; \mathbb{R}^n)$ to a map $v_\rho \in BD(P_\rho^x)$ and $Eu_{P_\rho^x, x, r_i}$ converge weakly* to Ev_ρ in $\mathcal{M}(P_\rho^x, \mathbb{R}^{n \times n})$, up to a subsequence not relabeled. Therefore, we find $Ev_\rho = \frac{|\mathrm{e}_1 \odot \mathrm{e}_2|}{|\eta \odot \xi|} Eg_\rho$ as measures on P_ρ^x (cf. Remark 3.12). In view of (3.5) in Theorem 3.4, there is an infinitesimal rigid motion such that

$$v_\rho(z) = \frac{|\mathrm{e}_1 \odot \mathrm{e}_2|}{|\eta \odot \xi|} g_\rho(z) + \mathbb{L}_\rho z + \mathrm{v}_\rho.$$

In addition, $|Ev_\rho|(P_\rho^x) = \mathcal{L}^n(P_\rho^x)$, so that in particular $u_{P_\rho^x, x, r_i}$ converge to v_ρ strictly in $BD(P_\rho^x)$. We complete the proof of (4.6) by deducing that $\mathfrak{R}_{P_\rho^x}[v_\rho] = \underline{0}$ from $\mathfrak{R}_{P_\rho^x}[u_{P_\rho^x, x, r_i}] = \underline{0}$ for all i and the above mentioned strict convergence in $BD(P_\rho^x)$. $\square$

Remark 4.3. Notice that the parallelogram P_ρ^x produced by Proposition 4.2, along which the sequences $u_{P_\rho^x, x, \varepsilon}$ converge strictly in $BD(P_\rho^x)$ to a function v_ρ affine in one direction, has been chosen in a way that the short edge (the edge of size ρ) is oriented exactly in the direction of nonaffinity of v_ρ. The nonaffine direction is exactly the one we need to control in order to conclude the representation Theorem 2.3, and thus the fact that the parallelograms P_ρ^x are well oriented plays a crucial role in the argument that will follow.

4.2. Finer analysis of the blow-up limits. We proceed next with the investigation of some properties of the blow-up limits provided by Proposition 4.2 that follow by exploiting their structure evidenced in (4.5)–(4.6). Similar results are available in the BD setting in case the base point is either a point of approximate differentiability or a jump point. The analogue of the ensuing result is also well-known for BV functions (see, for instance, [3, Theorem 3.95]).

We will state some technical lemmas that will allow us to identify in a more precise way the blow-up limits. To this aim, for a function $\psi \in BV((a, b))$ we denote by $\psi(a)$, $\psi(b)$ the right and left traces in $a, b \in \mathbb{R}$, respectively. We start with the case $\xi \neq \pm\eta$.

LEMMA 4.4. *Let* $\{v_\rho\}_{\rho > 0} \subseteq BV(P_\rho^{\xi, \eta}; \mathbb{R}^n)$, $\xi, \eta \in \mathbb{R}^n \setminus \{\underline{0}\}$ *with* $\xi \neq \pm\eta$, *be a sequence of functions such that*

$$v_\rho(x) = \bar\psi_\rho(x \cdot \eta)\xi + (x \cdot \xi)\bar\beta_\rho \, \eta + \mathbb{L}_\rho x + \mathrm{v}_\rho$$

for some $\bar\psi_\rho \in BV((-\rho/2, \rho/2)), \bar\beta_\rho \in \mathbb{R}, \mathrm{v}_\rho \in \mathbb{R}^n, \mathbb{L}_\rho \in \mathbb{M}_{skew}^{n \times n}$. *Assume also that*

$$\mathfrak{R}_{P_\rho^{\xi, \eta}}[v_\rho] = \underline{0}, \qquad |Ev_\rho| \left(P_\rho^{\xi, \eta}\right) = \mathcal{L}^n \left(P_\rho^{\xi, \eta}\right).$$

Then, v_ρ can be rewritten as

$$v_\rho(x) = \psi_\rho(x \cdot \eta)\xi + \frac{x \cdot \xi}{2|\eta \odot \xi|}\eta$$

for some $\psi_\rho \in BV((-\rho/2, \rho/2))$ with zero average such that $\{\psi_\rho(\cdot)/\rho\}_{\rho>0}$ is uniformly bounded in $L^\infty((-\rho/2, \rho/2))$ and

$$(4.9) \qquad D\psi_\rho\big((-\rho/2, \rho/2)\big) = \frac{\rho}{2|\eta \odot \xi|}\,.$$

Proof. Set $R_\rho := (-\rho/2, \rho/2) \times (-1/2, 1/2)^{n-1}$, and let $\mathbb{B} \in \mathbb{M}^{n\times n}$ be any invertible matrix such that $\mathbb{B}\eta = \mathrm{e}_1, \mathbb{B}\xi = \mathrm{e}_2$ and mapping R_ρ onto $P_\rho^{\xi,\eta}$. Namely, $P_\rho^{\xi,\eta} = \mathbb{B}^t R_\rho$. By invoking Lemma 3.11 and Remark 3.12 we can infer that

$$\tilde{v}_\rho(y) := \mathbb{B}v_\rho(\mathbb{B}^t y)$$

satisfies $\tilde{v}_\rho \in BD(R_\rho)$ and

$$\tilde{v}_\rho(y) = \bar{\psi}_\rho(y \cdot \mathrm{e}_1)\mathrm{e}_2 + (y \cdot \mathrm{e}_2)\bar{\beta}_\rho\,\mathrm{e}_1 + \tilde{\mathbb{L}}_\rho y + \tilde{\mathrm{v}}_\rho,$$

with $\tilde{\mathbb{L}}_\rho = \mathbb{B}\mathbb{L}_\rho\mathbb{B}^t$, $\tilde{\mathrm{v}}_\rho = \mathbb{B}\mathrm{v}_\rho$. Moreover $\mathfrak{R}_{R_\rho}[\tilde{v}_\rho] = \underline{0}$, and

$$(4.10) \qquad |E\tilde{v}_\rho|(R_\rho) = \frac{|\mathrm{e}_1 \odot \mathrm{e}_2|}{|\eta \odot \xi|}\rho, \qquad E\tilde{v}_\rho = \frac{\mathrm{e}_1 \odot \mathrm{e}_2}{|\mathrm{e}_1 \odot \mathrm{e}_2|}|E\tilde{v}_\rho|.$$

Step 1: Identification and properties of ψ_ρ and β_ρ. Condition $\mathfrak{R}_{R_\rho}[\tilde{v}_\rho] = \underline{0}$ is equivalent to $\mathbb{M}_{R_\rho}[\tilde{v}_\rho] = b_{R_\rho}[\tilde{v}_\rho] = \underline{0}$ (see (3.7) for the definition of b_{R_ρ} and (3.8) for that of $\mathbb{M}_{R_\rho}$). In turn, from these equalities we get that

$$\tilde{\mathrm{v}}_\rho + \left(\fint_{R_\rho} \bar{\psi}_\rho(y \cdot \mathrm{e}_1)\,\mathrm{d}y\right)\mathrm{e}_2 = \tilde{\mathrm{v}}_\rho + \left(\fint_{-\rho/2}^{\rho/2} \bar{\psi}_\rho(t)\,\mathrm{d}t\right)\mathrm{e}_2 = \underline{0}\,.$$

Thanks to Lemma 3.5 we get

$$\mathbb{M}_{R_\rho}\left[\left(\bar{\psi}_\rho(y \cdot \mathrm{e}_1) - \fint_{-\rho/2}^{\rho/2} \bar{\psi}_\rho(t)\,\mathrm{d}t\right)\mathrm{e}_2\right] = \frac{1}{2\rho}\big(\bar{\psi}_\rho(\rho/2) - \bar{\psi}_\rho(-\rho/2)\big)(\mathrm{e}_2 \otimes \mathrm{e}_1 - \mathrm{e}_1 \otimes \mathrm{e}_2)$$

and

$$\mathbb{M}_{R_\rho}\left[(y \cdot \mathrm{e}_2)\bar{\beta}_\rho\mathrm{e}_1\right] = \frac{\bar{\beta}_\rho}{2}(\mathrm{e}_1 \otimes \mathrm{e}_2 - \mathrm{e}_2 \otimes \mathrm{e}_1)\,.$$

Therefore, recalling that $\mathbb{M}_{R_\rho}[\tilde{\mathbb{L}}_\rho y] = \tilde{\mathbb{L}}_\rho y$, we conclude that for every $y \in R_\rho$

$$\tilde{\mathbb{L}}_\rho y + \mathbb{M}_{R_\rho}[\tilde{v}_\rho]y = \tilde{\mathbb{L}}_\rho y + \kappa_\rho(\mathrm{e}_2 \otimes \mathrm{e}_1 - \mathrm{e}_1 \otimes \mathrm{e}_2)y = \underline{0}\,,$$

where we have set

$$(4.11) \qquad \kappa_\rho := \frac{1}{2\rho}\big(\bar{\psi}_\rho(\rho/2) - \bar{\psi}_\rho(-\rho/2) - \bar{\beta}_\rho\rho\big).$$

In particular, by defining

$$\psi_\rho(t) := \bar{\psi}_\rho(t) - \kappa_\rho t - \fint_{-\rho/2}^{\rho/2} \bar{\psi}_\rho(t)\,\mathrm{d}t, \qquad \beta_\rho := \bar{\beta}_\rho + \kappa_\rho,$$

$\psi_\rho \in BV((-\rho/2, \rho/2))$ has zero average, and

(4.12) $$\tilde{v}_\rho(y) = \psi_\rho(x \cdot \mathrm{e}_1)\mathrm{e}_2 + (x \cdot \mathrm{e}_2)\beta_\rho \mathrm{e}_1.$$

Moreover, from the very definitions of $\tilde{v}_\rho$, ψ_ρ, and β_ρ we see that

$$E\tilde{v}_\rho = (\mathrm{e}_1 \odot \mathrm{e}_2)\left(D\bar{\psi}_\rho \otimes \mathcal{L}^{n-1} + \bar{\beta}_\rho \mathcal{L}^n\right) = (\mathrm{e}_1 \odot \mathrm{e}_2)\left(D\psi_\rho \otimes \mathcal{L}^{n-1} + \beta_\rho \mathcal{L}^n\right).$$

Let $I \subseteq (-\rho/2, \rho/2)$ be $\mathcal{L}^1$-measurable; then

$$\frac{\mathrm{e}_1 \odot \mathrm{e}_2}{|\mathrm{e}_1 \odot \mathrm{e}_2|}|E\tilde{v}_\rho|\left(I \times (-1/2, 1/2)^{n-1}\right) = E\tilde{v}_\rho\left(I \times (-1/2, 1/2)^{n-1}\right)$$

(4.13) $$= (\mathrm{e}_1 \odot \mathrm{e}_2)\left(D\bar{\psi}_\rho(I) + \bar{\beta}_\rho\mathcal{L}^1(I)\right) = (\mathrm{e}_1 \odot \mathrm{e}_2)\left(D\psi_\rho(I) + \beta_\rho\mathcal{L}^1(I)\right).$$

In particular, if $I = (-\rho/2, \rho/2)$ by exploiting (4.10) we conclude

(4.14) $$D\bar{\psi}_\rho\left((-\rho/2, \rho/2)\right) + \bar{\beta}_\rho\rho = D\psi_\rho\left((-\rho/2, \rho/2)\right) + \beta_\rho\rho = \frac{\rho}{|\eta \odot \xi|}.$$

Step 2: *The value of* β_ρ. Let $\mathcal{I} := \{I \subseteq (-\rho/2, \rho/2)\,\mathcal{L}^1\text{-measurable} : \mathcal{L}^1(I) > 0\}$; then from (4.13) we deduce that

$$\inf_{\mathcal{I}} \frac{D\bar{\psi}_\rho(I)}{\mathcal{L}^1(I)} + \bar{\beta}_\rho \geq 0, \qquad \inf_{\mathcal{I}} \frac{D\psi_\rho(I)}{\mathcal{L}^1(I)} + \beta_\rho \geq 0\ .$$

Hence, $t \mapsto \bar{\psi}_\rho(t) + \bar{\beta}_\rho t$ and $t \mapsto \psi_\rho(t) + \beta_\rho t$ are monotone nondecreasing functions.

Thus, thanks to the definition of κ_ρ we infer that $D\psi_\rho(-\rho/2, \rho/2) = \beta_\rho\rho$ as

$$D\psi_\rho\left((-\rho/2, \rho/2)\right) = D\bar{\psi}_\rho\left((-\rho/2, \rho/2)\right) - \kappa_\rho\rho$$
$$\overset{(4.11)}{=} \frac{1}{2}\left(D\bar{\psi}_\rho\left((-\rho/2, \rho/2)\right) + \bar{\beta}_\rho\rho\right) \overset{(4.14)}{=} \frac{1}{2}\left(D\psi_\rho\left((-\rho/2, \rho/2)\right) + \beta_\rho\rho\right).$$

In conclusion, we get

(4.15) $$D\psi_\rho\left((-\rho/2, \rho/2)\right) = \beta_\rho\rho = \frac{\rho}{2|\eta \odot \xi|}.$$

Step 3: *Inverse change of variables and conclusion.* By combining (4.12) and (4.15) we are thus led to

$$\tilde{v}_\rho(y) = \psi_\rho(x \cdot \mathrm{e}_1)\mathrm{e}_2 + \frac{(x \cdot \mathrm{e}_2)}{2|\eta \odot \xi|}\mathrm{e}_1\ .$$

Due to the very definition, $v_\rho(y) := \mathbb{B}^{-1}\tilde{v}_\rho(\mathbb{B}^{-t}y)$; thus we get that

$$v_\rho(y) = \psi_\rho(x \cdot \eta)\xi + \frac{x \cdot \xi}{2|\eta \odot \xi|}\eta\ .$$

Finally, by taking into account that $t \mapsto h_\rho(t) := \psi_\rho(t) + \beta_\rho t$ is monotone nondecreasing with zero average we get (by [3, Remark 3.50] and a simple scaling argument)

$$\|h_\rho\|_{L^\infty((-\rho/2, \rho/2))} \leq |Dh_\rho|\left((-\rho/2, \rho/2)\right) = \frac{\rho}{|\eta \odot \xi|}.$$

The statement for ψ_ρ then follows at once. $\qquad\square$

Similarly, we can characterize the case $\xi = \pm\eta$.

LEMMA 4.5. *Let* $\{v_\rho\}_{\rho>0} \subseteq BV(P_\rho^{\xi,\eta}; \mathbb{R}^n)$, $\xi, \eta \in \mathbb{R}^n \setminus \{\underline{0}\}$ *with* $\xi = \pm\eta$, *be a family of functions such that*

$$v_\rho(x) = \bar{\psi}_\rho(x \cdot \eta)\eta + \mathbb{L}_\rho x + \mathrm{v}_\rho$$

for some $\mathrm{v}_\rho \in \mathbb{R}^n$, $\mathbb{L}_\rho \in \mathbb{M}_{skew}^{n \times n}$, $\bar{\psi}_\rho \in BV\big((-\rho/2, \rho/2)\big)$. *Assume also that*

$$\mathfrak{R}_{P_\rho^{\xi,\eta}}[v_\rho] = \underline{0}, \qquad |Ev_\rho|\big(P_\rho^{\xi,\eta}\big) = \mathcal{L}^n\big(P_\rho^{\xi,\eta}\big) .$$

Then, v_ρ *can be rewritten as*

$$v_\rho(x) = \psi_\rho(x \cdot \eta)\eta$$

for a nondecreasing function $\psi_\rho \in BV((-\rho/2, \rho/2))$ *with zero average. Moreover,* $\{\psi_\rho(\cdot)/\rho\}_{\rho>0}$ *is uniformly bounded in* $L^\infty((-\rho/2, \rho/2))$, *and*

$$(4.16) \qquad |D\psi_\rho|\big((-\rho/2, \rho/2)\big) = \frac{\rho}{|\eta \odot \eta|} .$$

Let us summarize the results contained in Proposition 4.2, Lemma 4.4, and Lemma 4.5 in the following statement (see (4.1) for the definition of $u_{K,x,\varepsilon}$, and see (4.3), (4.4) for those of $P_\rho^{\xi,\eta}$).

PROPOSITION 4.6 (selecting a good blow-up $|E^c u|$-a.e.). *Let* $u \in BD(\Omega)$. *Then for* $|E^c u|$-*a.e.* $x \in \Omega$ *there exist vectors* $\xi, \eta \in \mathbb{S}^{n-1}$, $\{\rho_j\}_{j \in \mathbb{N}}$, $\rho_j \downarrow 0$ *as* $j \uparrow +\infty$, *and for all* $j \in \mathbb{N}$ *a sequence* $\{\varepsilon_{i,j}\}_{i \in \mathbb{N}}$, *with* $\varepsilon_{i,j} \downarrow 0$ *as* $i \uparrow +\infty$, *a bounded, open, convex set containing the origin* $P_j^x := P_{\rho_j}^{\xi,\eta}$ *and a function* v_j *such that* $u_{P_j^x, x, \varepsilon_{i,j}}$ *converge to* v_j *strictly in* $BD(P_j^x)$ *as* $i \uparrow +\infty$, *with*
 (a) *if* $\xi \neq \pm\eta$,

$$v_j(y) := \psi_j(y \cdot \eta)\xi + \frac{y \cdot \xi}{2|\eta \odot \xi|}\eta$$

 for some map $\psi_j \in BV\big((-\rho_j/2, \rho_j/2)\big)$ *with zero average such that*

$$D\psi_j\big((-\rho_j/2, \rho_j/2)\big) = \frac{\rho_j}{2|\eta \odot \xi|}, \qquad \sup_{j \in \mathbb{N}} \|\psi_j(\cdot)/\rho_j\|_{L^\infty((-\rho_j/2, \rho_j/2))} < +\infty;$$

 (b) *if* $\xi = \pm\eta$,

$$v_j(y) = \psi_j(y \cdot \eta)\eta$$

 for some nondecreasing map $\psi_j \in BV\big((-\rho_j/2, \rho_j/2)\big)$ *with zero average such that*

$$|D\psi_j|\big((-\rho_j/2, \rho_j/2)\big) = \frac{\rho_j}{|\eta \odot \eta|}, \qquad \sup_{j \in \mathbb{N}} \|\psi_j(\cdot)/\rho_j\|_{L^\infty((-\rho_j/2, \rho_j/2))} < +\infty.$$

Proof. We prove the result for the subset of points F for which Proposition 4.2 holds true. In particular, $|E^c u|(\Omega \setminus F) = 0$. One such point x being fixed, note that given any infinitesimal sequence $\{\rho_j\}_{j \in \mathbb{N}}$ we can extract a subsequence (not relabeled) along which the maps v_{ρ_j} provided by Proposition 4.2 are affine in one single direction (either η or ξ) provided in the statement. Without loss of generality we denote such a direction by ξ to be coherent with the notation of Proposition 4.2 itself.

Assume first $\xi \neq \pm\eta$. Thanks to Proposition 4.2 we can find a sequence of scales $\{\varepsilon_{i,j}\}_{i \in \mathbb{N}}$ such that the rescaled maps converge strictly in $BD(P_j^x)$ to a map v_{ρ_j} as in the statement there. By using Lemma 4.4 we conclude.

Finally, if $\xi = \pm\eta$ we argue similarly by using Lemma 4.5 rather than Lemma 4.4. $\square$

5. Proof of the main result. We first recall the results in [24, Theorem 3.3, Remark 3.5]. The original statement concerns integral representation of the volume and jump energy densities of functionals satisfying (H1)–(H4) and a more stringent version of (H5) (cf. Remark 5.2) and for functions in the subspace $SBD(\Omega)$. In what follows we state the result for the full space $BD(\Omega)$. Indeed, the same proof works with no difference since it is obtained via the global method for relaxation, hinging on a blow-up argument and the characterization of the energy densities in terms of the Dirichlet cell formulas defining $\mathfrak{m}$. We notice that (H4) and (H5) are actually not needed for the integral representation of the bulk and surface terms of the energy.

LEMMA 5.1. *Let $\mathcal{F}$ satisfy* (H1)–(H3). *Then, for every $u \in BD(\Omega)$*
(a) *for $\mathcal{L}^n$-a.e. $x_0 \in \Omega$*

$$\lim_{\varepsilon \to 0} \frac{\mathcal{F}(u, Q(x_0, \varepsilon))}{\varepsilon^n} = f(x_0, u(x_0), \nabla u(x_0)),$$

where f denotes the function defined in (2.5);
(b) *for $\mathcal{H}^{n-1}$-a.e. $x_0 \in J_u$*

$$\lim_{\varepsilon \to 0} \frac{\mathcal{F}(u, Q^{\nu_u(x_0)}(x_0, \varepsilon))}{\varepsilon^{n-1}} = g(x_0, u^-(x_0), u^+(x_0), \nu_u(x_0)),$$

where g denotes the function defined in (2.6),

It should be noted that Lemma 5.1 and the lower semicontinuity of the integral on $W^{1,1}$ imply that the Borel functions f and g are respectively quasiconvex (see [3, Definition 5.25]; see also formulas (5.3), (5.4) below) and BV elliptic (see [3, Definition 5.13]).

By taking into account (H2), we conclude that there exists a constant $C > 0$ such that for every $(x, \mathrm{v}, \mathbb{A}) \in \Omega \times \mathbb{R}^n \times \mathbb{M}^{n \times n}$

$$(5.1) \qquad \frac{1}{C}\Big|\frac{\mathbb{A} + \mathbb{A}^t}{2}\Big| \le f(x, \mathrm{v}, \mathbb{A}) \le C\Big(1 + \Big|\frac{\mathbb{A} + \mathbb{A}^t}{2}\Big|\Big),$$

and that for every $(x, \mathrm{v}^-, \mathrm{v}^+) \in \Omega \times (\mathbb{R}^n)^2$

$$\frac{1}{C}|(\mathrm{v}^+ - \mathrm{v}^-) \odot \nu| \le g(x, \mathrm{v}^-, \mathrm{v}^+, \nu) \le C|(\mathrm{v}^+ - \mathrm{v}^-) \odot \nu|.$$

Several other properties of f and g can be inferred according to the invariance properties satisfied by the functional $\mathcal{F}$ (cf. [7, Remark 3.8]). For instance, assumption (H5) implies that f depends only on the symmetric part of the relevant matrix. Indeed, from (3.18) we immediately deduce, for all $(x_0, \mathrm{v}_0, \mathbb{A}) \in \Omega \times \mathbb{R}^n \times \mathbb{M}^{n \times n}$ (thanks also to item (a) in Lemma 5.1), that

$$f(x_0, \mathrm{v}_0, \mathbb{A}) = \lim_{\varepsilon \to 0} \frac{\mathfrak{m}(\mathrm{v}_0 + \mathbb{A}(\cdot - x_0), Q(x_0, \varepsilon))}{\varepsilon^n}$$

$$(5.2) \qquad = \lim_{\varepsilon \to 0} \frac{\mathfrak{m}\left(\mathrm{v}_0 + \dfrac{\mathbb{A} + \mathbb{A}^t}{2}(\cdot - x_0), Q(x_0, \varepsilon)\right)}{\varepsilon^n} = f\left(x_0, \mathrm{v}_0, \frac{\mathbb{A} + \mathbb{A}^t}{2}\right).$$

Therefore, in this case we deduce that f is symmetric quasiconvex. Namely, for every bounded open set $D \subset \mathbb{R}^n$, for $\mathcal{L}^n$ a.e. $x \in \Omega$, for all $(\mathrm{v}, \mathbb{A}) \in \mathbb{R}^n \times \mathbb{M}^{n \times n}_{sym}$ and for all $\varphi \in \mathcal{C}^1_c(D; \mathbb{R}^n)$

$$(5.3) \qquad f(x, \mathrm{v}, \mathbb{A}) \le \fint_D f(x, \mathrm{v}, \mathbb{A} + e(\varphi)(y))\, \mathrm{d}y,$$

or, equivalently, for all $\varphi \in \mathcal{C}^1(Q^\nu(\underline{0}, 1); \mathbb{R}^n)$ that are $Q^\nu(\underline{0}, 1)$-periodic, it holds that

$$(5.4) \qquad f(x, \mathrm{v}, \mathbb{A}) \le \fint_{Q^\nu(\underline{0},1)} f(x, \mathrm{v}, \mathbb{A} + e(\varphi)(y)) \, \mathrm{d}y.$$

Remark 5.2. If, in addition, we strengthen (H5) to

$$(5.5) \qquad \mathcal{F}(u + \mathbb{L}(\cdot - x_0) + \mathrm{v}, A) = \mathcal{F}(u, A)$$

for every $(u, A, \mathrm{v}, \mathbb{L}, x_0) \in BD(\Omega) \times \mathcal{O}(\Omega) \times \mathbb{R}^n \times \mathbb{M}^{n \times n}_{skew} \times \Omega$, then the cell formulas imply $f(x, \mathrm{v}, \frac{\mathbb{A}+\mathbb{A}^t}{2}) = f(x, \frac{\mathbb{A}+\mathbb{A}^t}{2})$ and $g(x, \mathrm{v}^-, \mathrm{v}^+, \nu) = g(x, \mathrm{v}^+ - \mathrm{v}^-, \nu)$.

5.1. Preliminary lemmas. We now exploit the result in section 4, in particular Proposition 4.6, to deduce the asymptotic behavior of the energy around $|E^c u|$-a.e. point along the same line developed in [7, Lemma 3.9]. We keep the notation introduced in Proposition 4.6 and to simplify it we set $\psi_j := \psi_{\rho_j}$ and $P_j^{x_0} := P_{\rho_j}^{x_0}$ (for the definition of $\mathfrak{R}_K$ see (3.6)).

LEMMA 5.3. *Let $\mathcal{F}$ satisfy* (H1)–(H4). *Then, for every $u \in BD(\Omega)$ and for $|E^c u|$-a.e $x_0 \in \Omega$ there exist a sequence $\{\rho_j\}_{j \in \mathbb{N}}$ infinitesimal as $j \uparrow +\infty$, and for all $j \in \mathbb{N}$ an infinitesimal sequence $\{\varepsilon_{i,j}\}_{i \in \mathbb{N}}$ as $i \uparrow +\infty$, such that*

$$(5.6) \qquad \frac{\mathrm{d}\mathcal{F}(u, \cdot)}{\mathrm{d}|Eu|}(x_0) = \lim_{j \to +\infty} \limsup_{i \to +\infty} \frac{\mathfrak{m}\left(w_{i,j}, P_j^{x_0}(x_0, \varepsilon_{i,j})\right)}{|Eu|(P_j^{x_0}(x_0, \varepsilon_{i,j}))},$$

where

$$(5.7) \qquad w_{i,j}(y) := \mathfrak{R}_{P_j^{x_0}(x_0, \varepsilon_{i,j})}[u](y - x_0) + \frac{|Eu|(P_j^{x_0}(x_0, \varepsilon_{i,j}))}{\mathcal{L}^n(P_j^{x_0}(x_0, \varepsilon_{i,j})))} \frac{\eta \odot \xi}{|\eta \odot \xi|}(y - x_0).$$

Proof. We consider the subset of points of C_u for which Proposition 4.2 (and hence Proposition 4.6) is valid. For one such point $x_0 \in C_u$ consider the corresponding vectors ξ and $\eta \in \mathbb{S}^{n-1}$. Note that by Lemma 3.13 for any $j \in \mathbb{N}$

$$(5.8) \qquad \frac{\mathrm{d}\mathcal{F}(u, \cdot)}{\mathrm{d}|Eu|}(x_0) = \lim_{i \to +\infty} \frac{\mathfrak{m}(u, P_j^{x_0}(x_0, \varepsilon_{i,j}))}{|Eu|(P_j^{x_0}(x_0, \varepsilon_{i,j}))}.$$

Case 1: $\eta \ne \pm\xi$. By Proposition 4.6 we have that for every $j \in \mathbb{N}$

$$u_{i,j} := u_{P_j^{x_0}, x_0, \varepsilon_{i,j}} \to v_j := \psi_j(y \cdot \eta)\xi + \frac{y \cdot \xi}{2|\eta \odot \xi|}\eta \qquad \text{strict in } BD(P_j^{x_0}).$$

Define, for some constant c_j to be specified in what follows, the functions

$$(5.9) \qquad v_{i,j}(y) := w_{i,j}(y) + \varepsilon_{i,j}\frac{|Eu|(P_j^{x_0}(x_0, \varepsilon_{i,j}))}{\mathcal{L}^n(P_j^{x_0}(x_0, \varepsilon_{i,j}))}c_j\xi.$$

As by Remark 3.7

$$\mathfrak{R}_{P_j^{x_0}}[u(x_0 + \varepsilon_{i,j}\cdot)]\left(\frac{y - x_0}{\varepsilon_{i,j}}\right) = \mathfrak{R}_{P_j^{x_0}(x_0, \varepsilon_{i,j})}[u](y - x_0),$$

we have by Lemma 3.14

$$\frac{|\mathfrak{m}(u, P_j^{x_0}(x_0, \varepsilon_{i,j})) - \mathfrak{m}(v_{i,j}, P_j^{x_0}(x_0, \varepsilon_{i,j}))|}{|Eu|(P_j^{x_0}(x_0, \varepsilon_{i,j}))}$$

$$\leq C \int_{\partial P_j^{x_0}(x_0, \varepsilon_{i,j})} \frac{|u(y) - v_{i,j}(y)|}{|Eu|(P_j^{x_0}(x_0, \varepsilon_{i,j}))} \, d\mathcal{H}^{n-1}(y)$$

$$= \frac{C}{\rho_j} \int_{\partial P_j^{x_0}} \left| u_{i,j}(y) - \frac{\eta \odot \xi}{|\eta \odot \xi|} \, y - c_j \xi \right| d\mathcal{H}^{n-1}(y).$$

By taking the superior limit in i we thus get

$$\limsup_{i \to +\infty} \frac{|\mathfrak{m}(u, P_j^{x_0}(x_0, \varepsilon_{i,j})) - \mathfrak{m}(v_{i,j}, P_j^{x_0}(x_0, \varepsilon_{i,j}))|}{|Eu|(P_j^{x_0}(x_0, \varepsilon_{i,j}))}$$

$$\leq \frac{C}{\rho_j} \int_{\partial P_j^{x_0}} \left| \psi_j(y \cdot \eta) - \frac{(y \cdot \eta)}{2|\eta \odot \xi|} - c_j \right| d\mathcal{H}^{n-1}(y)$$

$$= \frac{C}{\rho_j} \left| \psi_j(\rho_j/2) - \frac{\rho_j}{4|\eta \odot \xi|} - c_j \right| + \frac{C}{\rho_j} \left| \psi_j(-\rho_j/2) + \frac{\rho_j}{4|\eta \odot \xi|} - c_j \right|$$

$$+ \frac{C}{\rho_j} \int_{-\rho_j/2}^{\rho_j/2} \left| \psi_j(t) - \frac{t}{2|\eta \odot \xi|} - c_j \right| dt,$$

recalling that $P_j^{x_0} = \{ y \in \mathbb{R}^n : |y \cdot \eta| \leq \rho_j/2, \, |y \cdot \xi| \leq 1/2, \, |y \cdot \zeta_i| \leq 1/2 \; i = 1, \ldots, n-2 \}$ and that ξ and η depend on x_0.

By choosing $c_j := \psi_j(-\rho_j/2) + \frac{\rho_j}{4|\eta \odot \xi|}$, since $\psi_j(\rho_j/2) - \psi_j(-\rho_j/2) = \frac{\rho_j}{2|\eta \odot \xi|}$, the first two summands in the last inequality are then null. Therefore, we have

$$\limsup_{i \to +\infty} \frac{|\mathfrak{m}(u, P_j^{x_0}(x_0, \varepsilon_{i,j})) - \mathfrak{m}(v_{i,j}, P_j^{x_0}(x_0, \varepsilon_{i,j}))|}{|Eu|(P_j^{x_0}(x_0, \varepsilon_{i,j}))}$$

$$\leq \frac{C}{\rho_j} \int_{-\rho_j/2}^{\rho_j/2} \left| \psi_j(t) - \frac{t}{2|\eta \odot \xi|} - c_j \right| dt \leq C \left(\|\psi_j\|_{L^\infty(-\frac{\rho_j}{2}, \frac{\rho_j}{2})} + \rho_j + c_j \right) \leq C\rho_j \,,$$

where we have used that $\psi_j(\cdot)/\rho_j$ is equi-bounded in $L^\infty((-\rho_j/2, \rho_j/2))$, and that $\psi_j(\pm\rho_j/2)$ are boundary trace values to infer $c_j \leq C\rho_j$, for some universal constant $C > 0$. In conclusion, we have proved that

$$\lim_{j \to +\infty} \limsup_{i \to +\infty} \frac{|\mathfrak{m}(u, P_j^{x_0}(x_0, \varepsilon_{i,j})) - \mathfrak{m}(v_{i,j}, P_j^{x_0}(x_0, \varepsilon_{i,j}))|}{|Eu|(P_j^{x_0}(x_0, \varepsilon_{i,j}))} = 0,$$

so that (5.8) yields

$$(5.10) \qquad \frac{d\mathcal{F}(u, \cdot)}{d|Eu|}(x_0) = \lim_{j \to +\infty} \limsup_{i \to +\infty} \frac{\mathfrak{m}(v_{i,j}, P_j^{x_0}(x_0, \varepsilon_{i,j}))}{|Eu|(P_j^{x_0}(x_0, \varepsilon_{i,j}))}.$$

Finally, recalling the definition of $w_{i,j}$ in (5.7) and of $v_{i,j}$ in (5.9), by estimate (3.17) we have

$$\left| \mathfrak{m}\left(v_{i,j}, P_j^{x_0}(x_0, \varepsilon_{i,j}) \right) - \mathfrak{m}(w_{i,j}, P_j^{x_0}(x_0, \varepsilon_{i,j})) \right|$$

$$\leq C_{P_j^{x_0}} \Psi \left(\varepsilon_{i,j} \frac{|Eu|(P_j^{x_0}(x_0, \varepsilon_{i,j}))}{\mathcal{L}^n(P_j^{x_0}(x_0, \varepsilon_{i,j}))} c_j \right) \left(\mathcal{L}^n(P_j^{x_0}(x_0, \varepsilon_{i,j})) + |Eu|(P_j^{x_0}(x_0, \varepsilon_{i,j})) \right),$$

and (5.6) then follows at once from (5.10) by letting $i \uparrow +\infty$, in view of the choice $x_0 \in C_u$.

Case 2: $\xi = \pm\eta$. Suppose, without loss of generality, that $\xi = \eta$. We argue as in Case 1. For the sequences $\{\rho_j\}_{j\in\mathbb{N}}$, $\{\varepsilon_{i,j}\}_{i\in\mathbb{N}}$ provided by Proposition 4.6 we have that

$$u_{P_j^{x_0}, x_0, \varepsilon_{i,j}} \to v_j := \psi_j(y \cdot \eta)\eta \qquad \text{strict in } BD(P_j^{x_0}).$$

By setting

$$v_{i,j}(y) := w_{i,j}(y) + \varepsilon_{i,j} \frac{|Eu|(P_j^{x_0}(x_0, \varepsilon_{i,j}))}{\mathcal{L}^n(P_j^{x_0}(x_0, \varepsilon_{i,j}))} c_j \eta,$$

for

$$c_j := \psi_j(-\rho_j/2) + \frac{\rho_j}{2|\eta \odot \eta|},$$

we conclude that

$$\lim_{j\to+\infty} \limsup_{i\to+\infty} \frac{\mathfrak{m}(v_{i,j}, P_j^{x_0}(x_0, \varepsilon_{i,j}))}{|Eu|(P_j^{x_0}(x_0, \varepsilon_{i,j}))} = \lim_{i\to+\infty} \frac{\mathfrak{m}(u, P_j^{x_0}(x_0, \varepsilon_{i,j}))}{|Eu|(P_j^{x_0}(x_0, \varepsilon_{i,j}))}.$$

We again combine this equality with (5.8) to conclude. $\qquad\square$

We now use assumption (H5) to prove a lower bound for the cell formula $\mathfrak{m}$ computed on affine functions as done in [7, Lemma 3.11].

LEMMA 5.4. *Let $\mathcal{F}$ satisfy* (H1)–(H5). *For all* $\mathrm{v} \in \mathbb{R}^n$, $\xi' \in \mathbb{R}^n \setminus \{\underline{0}\}$, $\eta \in \mathbb{S}^{n-1}$, $x_0 \in \Omega$ *and for every sequence* (t_i, ε_i) *such that $t_i \to +\infty$ and $\varepsilon_i t_i \to 0$, and for every $\rho > 0$, it holds that*

$$f(x_0, \mathrm{v}, \eta \odot \xi') - f(x_0, \mathrm{v}, 0) \le \liminf_{i\to+\infty} \frac{\mathfrak{m}(\mathrm{v} + t_i\, \eta \odot \xi'(\cdot - x_0), P_\rho^{\xi,\eta}(x_0, \varepsilon_i))}{t_i \mathcal{L}^n(P_\rho^{\xi,\eta}(x_0, \varepsilon_i))},$$

where $\xi := \xi'/|\xi'|$, $P_\rho^{\xi,\eta}$ is defined in either (4.3) or (4.4) according to whether $\xi \ne \pm\eta$ or not, and f is the volume energy density defined in item (a) *of Lemma* 5.1.

Proof. We start off noting that $\eta \otimes \xi' = \eta \odot \xi' + \mathbb{L}$, where $\mathbb{L} := \frac{1}{2}(\eta \otimes \xi' - \xi \otimes \eta') \in \mathbb{M}_{skew}^{n\times n}$. Then, in view of (H5) formula (3.18) implies

$$\left| \mathfrak{m}\left(\mathrm{v} + t_i\, \eta \otimes \xi'(\cdot - x_0), P_\rho^{\xi,\eta}(x_0, \varepsilon_i)\right) - \mathfrak{m}\left(\mathrm{v} + t_i\, \eta \odot \xi'(\cdot - x_0), P_\rho^{\xi,\eta}(x_0, \varepsilon_i)\right)\right|$$

$$(5.11) \qquad \le C_{P_\rho^{\xi,\eta}} \Psi(\varepsilon_i t_i |\mathbb{L}|)(1 + t_i |\xi'|)\varepsilon_i^n.$$

Hence,

$$\liminf_{i\to+\infty} \frac{\mathfrak{m}(\mathrm{v} + t_i\eta \odot \xi'(\cdot - x_0), P_\rho^{\xi,\eta}(x_0, \varepsilon_i))}{t_i \mathcal{L}^n(P_\rho^{\xi,\eta}(x_0, \varepsilon_i))} = \liminf_{i\to+\infty} \frac{\mathfrak{m}(\mathrm{v} + t_i\eta \otimes \xi'(\cdot - x_0), P_\rho^{\xi,\eta}(x_0, \varepsilon_i))}{t_i \mathcal{L}^n(P_\rho^{\xi,\eta}(x_0, \varepsilon_i))}$$

$$\ge f(x_0, \mathrm{v}, \eta \otimes \xi') - f(x_0, \mathrm{v}, 0).$$

For the last inequality we have used [7, Lemma 3.11]. Moreover, since by (5.2)

$$f(x_0, \mathrm{v}, \eta \otimes \xi') = f(x_0, \mathrm{v}, \eta \odot \xi'),$$

the conclusion follows at once. $\qquad\square$

Before proving Theorem 2.3, we note that the continuity estimate on $\mathfrak{m}$ contained in (3.17), deduced as a consequence of (H4), implies both

$$(5.12) \qquad |f(x_0 + x, \mathrm{v} + \mathrm{v}_0, \mathbb{A}) - f(x_0, \mathrm{v}_0, \mathbb{A})| \le \Psi(|x| + |\mathrm{v}|)(1 + |\mathbb{A}|)$$

and
$$|g(x_0 + x, \mathrm{v}_0 + \mathrm{v}, \mathrm{v}_1 + \mathrm{v}, \nu) - g(x_0, \mathrm{v}_0, \mathrm{v}_1, \nu)| \leq \Psi(|x| + |\mathrm{v}|)|\mathrm{v}_0 - \mathrm{v}_1|$$

for all $(x_0, x, \mathrm{v}_0, \mathrm{v}_1, \mathrm{v}, \nu, \mathbb{A}) \in (\Omega)^2 \times (\mathbb{R}^n)^3 \times \mathbb{S}^{n-1} \times \mathbb{M}_{sym}^{n \times n}$.

These properties are instrumental already in the BV setting to express the Radon–Nikodým derivative of $\mathcal{F}$ at u with respect to $|E^c u|$ in terms of an energy density computed on relevant quantities related to the base function u itself. In particular, by taking such properties into account, one can prove that the recession function f^∞ of the bulk energy density f is actually the energy density of the Cantor part.

5.2. Proof of the integral representation result.

Proof of Theorem 2.3. The representation of the volume and surface energy densities is dealt with in Lemma 5.1.

We then turn to the representation of the energy density of the Cantor part. For $|E^c u|$-a.e. point $x_0 \in C_u$ we may apply Lemma 5.3 (in what follows we keep the notation introduced there) and find infinitesimal sequences $\{\rho_j\}_{j \in \mathbb{N}}$, $\{\varepsilon_{i,j}\}_{i \in \mathbb{N}}$ such that

$$(5.13) \qquad \frac{\mathrm{d}\mathcal{F}(u, \cdot)}{\mathrm{d}|E^c u|}(x_0) = \frac{\mathrm{d}\mathcal{F}(u, \cdot)}{\mathrm{d}|Eu|}(x_0) = \lim_{j \to +\infty} \limsup_{i \to +\infty} \frac{\mathfrak{m}\left(w_{i,j}, P_j^{x_0}(x_0, \varepsilon_{i,j})\right)}{|Eu|(P_j^{x_0}(x_0, \varepsilon_{i,j}))}.$$

On setting

$$\mathrm{v}_{i,j} := \fint_{P_j^{x_0}(x_0, \varepsilon_{i,j})} u(x)\, \mathrm{d}x, \qquad \mathbb{L}_{i,j} := \mathbb{M}_{P_j^{x_0}(x_0, \varepsilon_{i,j})}[u]$$

and $t_{i,j} := \frac{|Eu|(P_j^{x_0}(x_0, \varepsilon_{i,j}))}{\mathcal{L}^n(P_j^{x_0}(x_0, \varepsilon_{i,j}))}$, we have $w_{i,j} = \mathrm{v}_{i,j} + \mathbb{L}_{i,j}(\cdot - x_0) + t_{i,j}\frac{\eta \odot \xi}{|\eta \odot \xi|}(\cdot - x_0)$. Note that $t_{i,j} \to +\infty$ as $i \to +\infty$ for all $j \in \mathbb{N}$ as $x_0 \in C_u$. Moreover, recall that x_0 is a point of approximate continuity of u.

Next we note that

$$\left|\mathfrak{m}\left(w_{i,j}, P_j^{x_0}(x_0, \varepsilon_{i,j})\right) - \mathfrak{m}\left(u(x_0) + t_{i,j}\frac{\eta \odot \xi}{|\eta \odot \xi|}(\cdot - x_0), P_j^{x_0}(x_0, \varepsilon_{i,j})\right)\right|$$

$$\leq \left|\mathfrak{m}\left(w_{i,j}, P_j^{x_0}(x_0, \varepsilon_{i,j})\right) - \mathfrak{m}\left(\mathrm{v}_{i,j} + t_{i,j}\frac{\eta \odot \xi}{|\eta \odot \xi|}(\cdot - x_0), P_j^{x_0}(x_0, \varepsilon_{i,j})\right)\right|$$

$$+ \left|\mathfrak{m}\left(\mathrm{v}_{i,j} + t_{i,j}\frac{\eta \odot \xi}{|\eta \odot \xi|}(\cdot - x_0), P_j^{x_0}(x_0, \varepsilon_{i,j})\right)\right.$$

$$\left. - \mathfrak{m}\left(u(x_0) + t_{i,j}\frac{\eta \odot \xi}{|\eta \odot \xi|}(\cdot - x_0), P_j^{x_0}(x_0, \varepsilon_{i,j})\right)\right|$$

$$\overset{(3.18), (3.17)}{\leq} C_{P_j^{x_0}}\left(\Psi(\varepsilon_{i,j}\mathrm{diam}(P_j^{x_0})|\mathbb{L}_{i,j}|) + \Psi(|\mathrm{v}_{i,j} - u(x_0)|)\right)(1 + t_{i,j})\varepsilon_{i,j}^n.$$

By taking into account $\mathrm{v}_{i,j} \to u(x_0)$ and $\varepsilon_{i,j}|\mathbb{L}_{i,j}| \to 0$ as $i \uparrow +\infty$ thanks to Lemma 3.10, the latter estimate combined with (5.13) leads to

$$\frac{\mathrm{d}\mathcal{F}(u, \cdot)}{\mathrm{d}|E^c u|}(x_0) = \frac{\mathrm{d}\mathcal{F}(u, \cdot)}{\mathrm{d}|Eu|}(x_0)$$

$$(5.14) \qquad = \lim_{j \to +\infty} \limsup_{i \to +\infty} \frac{\mathfrak{m}\left(u(x_0) + t_{i,j}\frac{\eta \odot \xi}{|\eta \odot \xi|}(\cdot - x_0), P_j^{x_0}(x_0, \varepsilon_{i,j})\right)}{\mathcal{L}^n(P_j^{x_0}(x_0, \varepsilon_{i,j}))t_{i,j}}.$$

With fixed $j \in \mathbb{N}$ and $\lambda > 0$, by applying Lemma 5.4 with $\xi' = \lambda\frac{\xi}{|\eta \odot \xi|}$ and $t_i := t_{i,j}/\lambda$, (5.14) implies

$$\frac{\mathrm{d}\mathcal{F}(u,\cdot)}{\mathrm{d}|E^c u|}(x_0) \geq \frac{f\left(x_0, u(x_0), \lambda\frac{\eta\odot\xi}{|\eta\odot\xi|}\right) - f(x_0, u(x_0), 0)}{\lambda}.$$

Hence, by taking the superior limit as $\lambda \uparrow +\infty$ we infer

$$(5.15) \qquad \frac{\mathrm{d}\mathcal{F}(u,\cdot)}{\mathrm{d}|E^c u|}(x_0) \geq f^\infty\left(x_0, u(x_0), \frac{\eta\odot\xi}{|\eta\odot\xi|}\right).$$

On the other hand, using as a competitor in the cell problem defining

$$\mathfrak{m}\Big(u(x_0) + t_{i,j}\frac{\eta\odot\xi}{|\eta\odot\xi|}(\cdot - x_0), P_j^{x_0}(x_0, \varepsilon_{i,j})\Big)$$

the affine map $u(x_0) + t_{i,j}\frac{\eta\odot\xi}{|\eta\odot\xi|}(\cdot - x_0)$ itself, we can apply item (a) in Lemma 5.1 to deduce

$$\frac{\mathfrak{m}\left(u(x_0) + t_{i,j}\frac{\eta\odot\xi}{|\eta\odot\xi|}(\cdot - x_0), P_j^{x_0}(x_0, \varepsilon_{i,j})\right)}{t_{i,j}\mathcal{L}^n(P_j^{x_0}(x_0, \varepsilon_{i,j}))} \leq \frac{\mathcal{F}\left(u(x_0) + t_{i,j}\frac{\eta\odot\xi}{|\eta\odot\xi|}(\cdot - x_0), P_j^{x_0}(x_0, \varepsilon_{i,j})\right)}{t_{i,j}\mathcal{L}^n(P_j^{x_0}(x_0, \varepsilon_{i,j}))}$$

$$= \fint_{P_j^{x_0}(x_0, \varepsilon_{i,j})} \frac{1}{t_{i,j}}f\left(x, u(x_0) + t_{i,j}\frac{\eta\odot\xi}{|\eta\odot\xi|}(x - x_0), t_{i,j}\frac{\eta\odot\xi}{|\eta\odot\xi|}\right)\mathrm{d}x$$

$$\overset{(5.12)}{\leq} \frac{1}{t_{i,j}}f\left(x_0, u(x_0), t_{i,j}\frac{\eta\odot\xi}{|\eta\odot\xi|}\right) + 2\Psi(\varepsilon_{i,j} + \varepsilon_{i,j}t_{i,j})$$

$$\leq \frac{1}{t_{i,j}}\left(f\left(x_0, u(x_0), t_{i,j}\frac{\eta\odot\xi}{|\eta\odot\xi|}\right) - f(x_0, u(x_0), 0)\right) + \frac{C}{t_{i,j}}.$$

Since, $t_{i,j} \to +\infty$ as $i \to +\infty$ for all $j \in \mathbb{N}$, we deduce that

$$\lim_{j\to+\infty}\limsup_{i\to+\infty}\frac{\mathfrak{m}\left(u(x_0) + t_{i,j}\frac{\eta\odot\xi}{|\eta\odot\xi|}(\cdot - x_0), P_j^{x_0}(x_0, \varepsilon_{i,j})\right)}{\mathcal{L}^n(P_j^{x_0}(x_0, \varepsilon_{i,j})))t_{i,j}} \leq f^\infty\left(x_0, u(x_0), \frac{\eta\odot\xi}{|\eta\odot\xi|}\right),$$

which combined with (5.14) and (5.15) finally leads to

$$\frac{\mathrm{d}\mathcal{F}(u,\cdot)}{\mathrm{d}|E^c u|}(x_0) = f^\infty\left(x_0, u(x_0), \frac{\mathrm{d}E^c u}{\mathrm{d}|E^c u|}(x_0)\right). \qquad\qquad \square$$

A standard monotone approximation technique provides the following extension of Theorem 2.3 (see section 7 for a similar argument).

COROLLARY 5.5. *Let $\mathcal{F} : L^1(\Omega; \mathbb{R}^n) \times \mathcal{O}(\Omega) \to [0, +\infty]$ satisfy* (H1), (H3), (H4), (H5) *and in place of* (H2), *we have the following:*
(HH2) *there exists a constant $C > 0$ such that for every $(u, A) \in BD(\Omega) \times \mathcal{O}(\Omega)$*

$$0 \leq \mathcal{F}(u, A) \leq C(\mathcal{L}^n(A) + |Eu|(A)).$$

Then, the conclusions of Theorem 2.3 *hold for all $u \in BD(\Omega)$.*

Proof. Let $\delta > 0$, and consider the functionals $\mathcal{F}_\delta : L^1(\Omega; \mathbb{R}^n) \times \mathcal{O}(\Omega) \to [0, +\infty]$ defined by $\mathcal{F}_\delta(u, A) := \mathcal{F}(u, A) + \delta|Eu|(A)$. Since $\mathcal{F}_\delta$ satisfies all the conditions (H1)–(H5) of Theorem 2.3 there are two functions f_δ and g_δ such that $\mathcal{F}_\delta$ can be represented as in the statement there. The family of functionals $\mathcal{F}_\delta$ is pointwise increasing in δ; therefore there exist the pointwise limits f of f_δ and g of g_δ as $\delta \downarrow 0$ (note that from the very definition of recession function $f_\delta^\infty(x, \mathrm{v}, \mathbb{A}) = f^\infty(x, \mathrm{v}, \mathbb{A}) + \delta|\mathbb{A}|)$. As $\mathcal{F}_\delta(\cdot, A)$ is pointwise converging to $\mathcal{F}(\cdot, A)$ for $\delta \downarrow 0$ on $BD(\Omega)$ for all $A \in \mathcal{O}(\Omega)$, we

conclude that the integral representation with energy densities f, g, and f^∞ for the bulk, surface, and Cantor parts, respectively, holds for $\mathcal{F}$. □

6. Some applications. Following [7, section 4] we provide some applications of the integral representation Theorem 2.3 to the topics of relaxation of bulk energies, of bulk and interfacial energies, to that of L^1 lower semicontinuity of functionals defined on BD, and to that of homogenization of bulk energies.

Throughout the section, for all $\nu \in \mathbb{S}^{n-1}$ and $r > 0$ we denote the cubes $Q^\nu(\underline{0}, r)$ by Q_r^ν, and moreover $Q(\underline{0}, r)$ by Q_r.

6.1. Relaxation and L^1 lower semicontinuity of bulk energies. In this section we address the issue of giving an explicit expression to the L^1 lower semicontinuous envelope of a linearly growing functional defined on smooth maps, for instance, $LD(\Omega)$.

Theorem 6.1 below generalizes to $BD(\Omega)$ the results proven in [5] and [24] on $SBD(\Omega)$. In particular, in [5] a continuous autonomous integrand f_0 (i.e., depending only on the symmetric gradient) is considered; the integral representation is then given in terms of the symmetric quasiconvex envelope of f_0 (see the definition below) and its associated recession function. In addition, Theorem 6.1 also generalizes partially the results on $BD(\Omega)$ established in [42] and [4, Corollary 1.10], [35, Theorem 1.4]. Note that in the former, also a Dirichlet boundary condition is considered, while in the last two integral representations of the weakly* lower semicontinuous envelope of functionals with linear growth at infinity are provided for more general PDEs constraints on the approximating sequences.

We stress that in the ensuing result, the strong recession function is not required to exist and that the integrand is allowed to depend also on x and v. Moreover, global continuity is replaced by the weaker condition (H2′).

Let $f_0 : \Omega \times \mathbb{R}^n \times \mathbb{M}_{sym}^{n \times n} \to [0, +\infty)$ be a Borel integrand satisfying the following:

(H1′) There exists a constant $C > 0$ such that for every $(x, \mathrm{v}, \mathbb{A}) \in \Omega \times \mathbb{R}^n \times \mathbb{M}_{sym}^{n \times n}$

$$(6.1) \qquad \frac{1}{C}|\mathbb{A}| \le f_0(x, \mathrm{v}, \mathbb{A}) \le C\big(1 + |\mathbb{A}|\big).$$

(H2′) There exists a constant $C > 0$ such that for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$|x - x_0| + |\mathrm{v} - \mathrm{v}_0| \le \delta \Longrightarrow |f_0(x, \mathrm{v}, \mathbb{A}) - f_0(x_0, \mathrm{v}_0, \mathbb{A})| \le C\varepsilon(1 + |\mathbb{A}|)$$

for every $(x, \mathrm{v}, \mathrm{v}_0, \mathbb{A}) \in \Omega \times (\mathbb{R}^n)^2 \times \mathbb{M}_{sym}^{n \times n}$.

Let then $\mathcal{F}_0 : L^1(\Omega; \mathbb{R}^n) \times \mathcal{O}(\Omega) \to [0, +\infty]$ be the functional defined by

$$(6.2) \qquad \mathcal{F}_0(u, A) := \inf \left\{ \liminf_{j \to +\infty} F_0(u_j, A) : u_j \to u \text{ in } L^1(\Omega; \mathbb{R}^n) \right\},$$

namely the $L^1(\Omega; \mathbb{R}^n)$ lower semicontinuous envelope of the functional

$$(6.3) \qquad F_0(u, A) := \begin{cases} \displaystyle\int_A f_0\big(x, u(x), e(u)(x)\big) \, \mathrm{d}x & \text{if } u \in LD(\Omega), \\[2mm] +\infty & \text{otherwise on } L^1(\Omega; \mathbb{R}^n). \end{cases}$$

We denote by $\mathfrak{m}$ the cell formula defined in (2.4) and related to $\mathcal{F}_0$, and recall the notation $u_{\mathrm{v}^-, \mathrm{v}^+, \nu}$ introduced in (2.8). We also recall that f^∞ stands for the (weak) recession function as defined by (2.7).

THEOREM 6.1. *Let $F_0 : L^1(\Omega; \mathbb{R}^n) \times \mathcal{O}(\Omega) \to [0, +\infty]$ be the functional defined in* (6.3). *Then, assuming* (H1′)–(H2′), *the functional* $\mathcal{F}_0 : L^1(\Omega; \mathbb{R}^n) \times \mathcal{O}(\Omega) \to [0, +\infty]$ *defined in* (6.2) *is represented by*

$$\mathcal{F}_0(u, A) = \int_A f\big(x, u(x), e(u)(x)\big) \, \mathrm{d}x$$
$$+ \int_{J_u \cap A} g\big(x, u^-(x), u^+(x), \nu_u(x)\big) \, \mathrm{d}\mathcal{H}^{n-1}(x)$$
$$+ \int_A f^\infty\Big(x, u(x), \frac{\mathrm{d}E^c u}{\mathrm{d}|E^c u|}(x)\Big) \, \mathrm{d}|E^c u|(x)$$

for all $(u, A) \in BD(\Omega) \times \mathcal{O}(\Omega)$, *where for every* $(x_0, \mathrm{v}, \mathbb{A}) \in \Omega \times \mathbb{R}^n \times \mathbb{M}^{n \times n}_{sym}$

$$(6.4) \qquad f(x_0, \mathrm{v}, \mathbb{A}) = \limsup_{\varepsilon \to 0} \inf_{\substack{w \in LD(Q_1) \\ w|_{\partial Q_1} = \mathbb{A}y|_{\partial Q}}} \int_{Q_1} f_0\big(x_0, \mathrm{v} + \varepsilon w(y), e(w)(y)\big) \, \mathrm{d}y,$$

and for every $(x_0, \mathrm{v}^-, \mathrm{v}^+, \nu) \in \Omega \times (\mathbb{R}^n)^2 \times \mathbb{S}^{n-1}$

$$(6.5) \quad g(x_0, \mathrm{v}^-, \mathrm{v}^+, \nu) = \limsup_{\varepsilon \to 0} \inf_{\substack{w \in LD(Q_1^\nu) \\ w|_{\partial Q_1^\nu} = u_{\mathrm{v}^-, \mathrm{v}^+, \nu}|_{\partial Q_1^\nu}}} \int_{Q_1^\nu} \varepsilon \, f_0\left(x_0, w(y), \frac{1}{\varepsilon} e(w)(y)\right) \, \mathrm{d}y.$$

Remark 6.2. Assumption (H2′) implies that $\mathcal{F}_0$ satisfies (H4). Instead, in the BV setting, the ensuing weaker assumption (H3′) replaces (H2′) in [7, section 4.1]:

(H3′) There exists a constant $C > 0$ such that for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$|\mathrm{v} - \mathrm{v}_0| \le \delta \Longrightarrow |f_0(x, \mathrm{v}, \mathbb{A}) - f_0(x, \mathrm{v}_0, \mathbb{A})| \le C\varepsilon(1 + |\mathbb{A}|)$$

for every $(x, \mathrm{v}, \mathrm{v}_0, \mathbb{A}) \in \Omega \times (\mathbb{R}^n)^2 \times \mathbb{M}^{n \times n}_{sym}$.

The latter condition and a truncation argument (cf. [7, Lemma 2.6]) are employed both to simplify the minimum problems defining f and g with respect to the v-variable (cf. [7, (4.1.5) and (4.1.6)] and to dispense with the analogue of (H4). Therefore, since the integral representation result in the BV setting cannot be directly applied, the quoted truncation argument and the analogue of Lemma 5.3 are central to give an explicit formula for the energy density of the Cantor part of the relaxed functionals analogous to $\mathcal{F}_0$ (cf. [7, (4.1.7) in Theorem 4.1.4] and [7, Remark 4.1.5]). Instead, since truncations are not permitted in the current BD setting, we need the stronger assumption (H2′) to enforce (H4).

Remark 6.3. In order to prove the integral representation of $\mathcal{F}_0$ over the subspace $SBD(\Omega)$ only, assumption (H2′) is actually not needed and (H1′) can be weakened. Indeed, to that aim one can allow for f_0 to depend also on the skew-symmetric part of the given matrix in view of Lemma 5.1 (cf. section 7). Clearly, formulas (6.4) defining f and (6.5) defining g have to be changed accordingly.

Remark 6.4. In view of the density of $W^{1,1}(\Omega; \mathbb{R}^n)$ in $BD(\Omega)$ with respect to the strict topology with assigned boundary trace (cf. [5, Theorem 2.6]), the space LD can be substituted with $W^{1,1}$ in the minimum problems defining f and g.

Therefore, the same conclusions of Theorem 6.1 can be drawn if we consider the functional $F'_0(u, A) := F_0(u, A)$ for $u \in W^{1,1}(\Omega; \mathbb{R}^n)$, and $+\infty$ otherwise on $L^1(\Omega; \mathbb{R}^n)$. Then the space of test maps for the minimum problems defining f and g in (6.4) and (6.5), respectively, is $W^{1,1}(\Omega; \mathbb{R}^n)$.

The main steps to prove Theorem 6.1 are similar to those exploited for the analogous result in the BV setting in [7, section 4.1] to which we refer. Therefore, we provide only a sketch of those proofs for which some changes are needed.

LEMMA 6.5 (see [7, Lemma 4.1.3]). *Assume* (H1$'$)–(H2$'$). *Then,* $\mathcal{F}_0$ *satisfies* (H1)–(H5)*, and for all* $(u, A) \in BD(\Omega) \times \mathcal{O}_\infty(\Omega)$

$$\mathfrak{m}(u, A) = \mathfrak{m}_0(u, A) := \inf\{F_0(w, A) : w \in BD(A), w = u \text{ on } \partial A\}.$$

Proof. Assumptions (H1), (H2), and (H5) are easily checked to be satisfied by $\mathcal{F}_0$ in view of (H1$'$) and the very definition of $\mathcal{F}_0$. Instead, (H4) follows from (H2$'$). For what concerns (H3) one can argue similarly to [5, Proposition 3.9].

The inequality $\mathfrak{m}(u, A) \leq \mathfrak{m}_0(u, A)$ follows from $\mathcal{F}_0 \leq F_0$. For the opposite, one uses the very definition of $\mathcal{F}_0$ as the relaxation of F_0 together with the version of the De Giorgi's averaging/slicing lemma stated in [5, Proposition 3.7]. □

By means of the alternative characterization of $\mathfrak{m}$ provided by $\mathfrak{m}_0$, of Theorem 2.3, of the results in section 4, and of a change of variable, Theorem 6.1 follows at once.

If an additional quantified closeness condition between f computed on large gradients and f^∞ is added, formula (6.5) defining g can be simplified. The ensuing claim (6.6) is well-known in the BV case (cf. [7, Theorem 4.1.4]).

COROLLARY 6.6. *Under the assumptions and notation of Theorem* 6.1 *and in addition*

(H4$'$) *on setting for any* $L > 0$ *and* $x_0 \in \Omega$

$$\omega_{f_0}(x_0, L) := \sup_{\substack{\mathrm{v} \in \mathbb{R}^n,\, t \geq L \\ \mathbb{A} \in \mathbb{M}^{n \times n}_{sym} :\, |\mathbb{A}| = 1}} \left| f_0^\infty(x_0, \mathrm{v}, \mathbb{A}) - \frac{1}{t} f_0(x_0, \mathrm{v}, t\mathbb{A}) \right|,$$

then $\omega_{f_0}(x_0, L)$ *is infinitesimal as* $L \to +\infty$,
the function g *in* (6.5) *in the conclusions of Theorem* 6.1 *is characterized alternatively by*

$$(6.6) \qquad g(x_0, \mathrm{v}^-, \mathrm{v}^+, \nu) = \inf_{\substack{w \in LD(Q_1^\nu) \\ w|_{\partial Q_1^\nu} = u_{\mathrm{v}^-, \mathrm{v}^+, \nu}|_{\partial Q_1^\nu}}} \int_{Q_1^\nu} f_0^\infty\big(x_0, w(y), e(w)(y)\big) \, \mathrm{d}y.$$

Furthermore, we deal with the v-independent case for which (H4$'$) is actually not needed (cf. [28, Remark 2.17] for the analogous result in the BV setting). We start off with a preliminary result.

COROLLARY 6.7. *Under the assumptions and notation of Theorem* 6.1*, if the integrand* f_0 *satisfies* $f_0(x, \mathrm{v}, \mathbb{A}) = f_0(x, \mathbb{A})$ *for every* $(x, \mathrm{v}, \mathbb{A}) \in \Omega \times \mathbb{R}^n \times \mathbb{M}^{n \times n}_{sym}$, *then*

$$g(x_0, \mathrm{v}^-, \mathrm{v}^+, \nu) = f^\infty\big(x_0, (\mathrm{v}^+ - \mathrm{v}^-) \odot \nu\big)$$

for every $(x_0, \mathrm{v}^-, \mathrm{v}^+, \nu) \in \Omega \times (\mathbb{R}^n)^2 \times \mathbb{S}^{n-1}$.

Proof. We start off defining

$$\widetilde{F}_0(u, A) := \begin{cases} \displaystyle\int_A f\big(x, e(u)(x)\big) \, \mathrm{d}x & \text{if } u \in LD(\Omega), \\ +\infty & \text{otherwise on } L^1(\Omega; \mathbb{R}^n), \end{cases}$$

where f is given by (6.4). Note that f is v-independent as $\mathcal{F}_0(u + \mathrm{z}, A) = \mathcal{F}_0(u, A)$ for all $(u, A, \mathrm{z}) \in BD(\Omega) \times \mathcal{O}(\Omega) \times \mathbb{R}^n$. Moreover, $f(x_0, \mathbb{A}) \leq f_0(x_0, \mathbb{A})$, for all $(x_0, \mathbb{A}) \in \Omega \times \mathbb{M}^{n \times n}_{sym}$, by using the linear function $\mathbb{A}y$ itself as a test in (6.4) and (H2′).

Denote by $\widetilde{\mathcal{F}}_0$ the $L^1(\Omega; \mathbb{R}^n)$ lower semicontinuous envelope of $\widetilde{F}_0$. Then, $\widetilde{F}_0 \leq F_0$ implies that $\widetilde{\mathcal{F}}_0 \leq \mathcal{F}_0$ on $L^1(\Omega; \mathbb{R}^n)$, and since $\mathcal{F}_0 = \widetilde{F}_0$ on $LD(\Omega)$ and $\mathcal{F}_0 \leq \widetilde{F}_0$ otherwise, we conclude that $\mathcal{F}_0 \equiv \widetilde{\mathcal{F}}_0$. Therefore, equality (6.5) defining g holds with f in place of f_0 in the minimum problem there. In passing, we point out that the invariance of f_0 implies that $g = g(x, \mathrm{v}^+ - \mathrm{v}^-, \nu)$, as well (cf. Remark 5.2).

Let us first prove that $g(x_0, \mathrm{v}^+ - \mathrm{v}^-, \nu) \geq f^\infty\big(x_0, (\mathrm{v}^+ - \mathrm{v}^+) \odot \nu\big)$. Let $\{\nu_1, \ldots, \nu_{n-1}, \nu\}$ form an orthonormal basis of $\mathbb{R}^n$ and set

$$\mathcal{A}^\nu := \big\{w \in W^{1,1}(Q_1^\nu; \mathbb{R}^n) : w|_{\{x \in \partial Q_1^\nu : x \cdot \nu = \pm 1/2\}} = 0,$$
$$w \text{ is 1-periodic in } \nu_i, \ 1 \leq i \leq n - 1\big\},$$

and

$$\mathcal{B}^\nu := \big\{w \in W^{1,1}(Q_1^\nu; \mathbb{R}^n) : (w - u_{\mathrm{v}^-, \mathrm{v}^+, \nu})|_{\{x \in \partial Q_1^\nu : x \cdot \nu = \pm 1/2\}} = 0,$$
$$w \text{ is 1-periodic in } \nu_i, \ 1 \leq i \leq n - 1\big\}.$$

In particular, note that $\mathcal{B}^\nu = \zeta(\cdot) + \mathcal{A}^\nu$, where

$$(6.7) \qquad \zeta(x) := \frac{\mathrm{v}^+ + \mathrm{v}^-}{2} + (\mathrm{v}^+ - \mathrm{v}^-)x \cdot \nu.$$

We then argue as follows:

$$g(x_0, \mathrm{v}^+ - \mathrm{v}^-, \nu) = \limsup_{\varepsilon \to 0} \inf_{\substack{w \in LD(Q_1^\nu) \\ w|_{\partial Q_1^\nu} = u_{\mathrm{v}^-, \mathrm{v}^+, \nu}|_{\partial Q_1^\nu}}} \int_{Q_1^\nu} \varepsilon f\left(x_0, \frac{1}{\varepsilon} e(w)(y)\right) \, \mathrm{d}y$$

$$= \limsup_{\varepsilon \to 0} \inf_{\substack{w \in W^{1,1}(Q_1^\nu; \mathbb{R}^n) \\ w|_{\partial Q_1^\nu} = u_{\mathrm{v}^-, \mathrm{v}^+, \nu}|_{\partial Q_1^\nu}}} \int_{Q_1^\nu} \varepsilon f\left(x_0, \frac{1}{\varepsilon} e(w)(y)\right) \, \mathrm{d}y$$

$$\geq \limsup_{\varepsilon \to 0} \inf_{w \in \mathcal{B}^\nu} \int_{Q_1^\nu} \varepsilon f\left(x_0, \frac{1}{\varepsilon} e(w)(y)\right) \, \mathrm{d}y$$

$$(6.8) \qquad = \limsup_{\varepsilon \to 0} \inf_{w \in \mathcal{A}^\nu} \int_{Q_1^\nu} \varepsilon f\left(x_0, \frac{1}{\varepsilon}(\mathrm{v}^+ - \mathrm{v}^-) \odot \nu + \frac{1}{\varepsilon} e(w)(y)\right) \, \mathrm{d}y \, ,$$

where for the second equality we have used Remark 6.4, and for the last, (6.7). Using the characterization of symmetric quasiconvexity expressed in (5.4) we conclude from (6.8) that

$$g(x_0, \mathrm{v}^+ - \mathrm{v}^-, \nu) \geq \limsup_{\varepsilon \to 0} \varepsilon f\left(x_0, \frac{1}{\varepsilon}(\mathrm{v}^+ - \mathrm{v}^-) \odot \nu\right) = f^\infty\big(x_0, (\mathrm{v}^+ - \mathrm{v}^-) \odot \nu\big).$$

To prove the opposite inequality, consider the affine function ζ in (6.7), extend it by 1-periodicity in the directions ν_i, $1 \leq i \leq n - 1$, and extend it further by $\mathrm{v}^\pm$ if $\pm x \cdot \nu \geq 1/2$ (with a slight abuse we keep the same notation for the extended function). Next let $\widetilde{w} \in W^{1,1}(Q_1^\nu; \mathbb{R}^n)$ have the same trace of $u_{\mathrm{v}^-, \mathrm{v}^+, \nu}$ on ∂Q_1^ν (cf. [5, Theorem 2.6]), extend it by 1-periodicity in the directions ν_i, $1 \leq i \leq n - 1$, and then extend it

by $u_{\mathrm{v}^-,\mathrm{v}^+,\nu}$ on the complement set (again we keep the same notation for the extended function). Let r and $\delta \in (0,1)$ and fix a cut-off function $\varphi \in W_0^{1,\infty}(Q_1^\nu;[0,1])$ such that $\varphi|_{Q_r^\nu} \equiv 1$. Define $\zeta_{r,\delta} := \varphi\,\zeta(\cdot/\delta) + (1-\varphi)\widetilde{w}(\cdot/\delta) \in W^{1,1}(Q_1^\nu;\mathbb{R}^n)$ and use it as a test function in the minimum problem in (6.5). Note that $\zeta_{r,\delta} = \mathrm{v}^\pm$ if $\pm x \cdot \nu \geq 1/2$. A simple computation yields,

$$e(\zeta_{r,\delta}) = \frac{1}{\delta}\big(\varphi\, e(\zeta)(\cdot/\delta) + (1-\varphi)e(\widetilde{w})(\cdot/\delta)\big) + \nabla\varphi \odot (\zeta(\cdot/\delta) - \widetilde{w}(\cdot/\delta));$$

thus using (H1$'$), a simple scaling argument and periodicity give for some $\omega_\delta \downarrow 0$ as $\delta \downarrow 0$

$$g(x_0,\mathrm{v}^+ - \mathrm{v}^-,\nu) \leq \limsup_{\varepsilon \to 0} \int_{Q_1^\nu} \varepsilon f\left(x_0, \frac{1}{\varepsilon}e(\zeta_{r,\delta})(y)\right)\,\mathrm{d}y$$

$$= \limsup_{\varepsilon \to 0} \left(\int_{Q_1^\nu \cap \{|x\cdot\nu| \leq \delta/2\}} \varepsilon f\left(x_0, \frac{1}{\varepsilon}e(\zeta_{r,\delta})(y)\right)\,\mathrm{d}y + \varepsilon f\big(x_0,0\big)\right)$$

$$\leq (1+\omega_\delta)\limsup_{\varepsilon \to 0} \int_{Q_1^\nu} \varepsilon\delta f\left(x_0, \frac{1}{\varepsilon\delta}(\mathrm{v}^+ - \mathrm{v}^+) \odot \nu\right)\,\mathrm{d}y$$

$$+ C(1+\omega_\delta)\int_{(Q_1^\nu \setminus Q_r^\nu) \cap \{|x\cdot\nu| \leq \delta/2\}} \left(\frac{1}{\delta}|e(\zeta)(y/\delta)| + \frac{1}{\delta}|e(\widetilde{w})(y/\delta)|\right.$$

$$\left. + |\nabla\varphi||\zeta(y/\delta) - \widetilde{w}(y/\delta)|\right)\,\mathrm{d}y$$

$$\leq (1+\omega_\delta)f^\infty\big(x_0,(\mathrm{v}^+ - \mathrm{v}^-)\odot\nu\big) + C\int_{Q_1^\nu \setminus Q_r^\nu} (|e(\zeta)| + |e(\widetilde{w})|)\,\mathrm{d}y$$

$$+ \frac{C\delta}{1-r}\int_{Q_1^\nu \setminus Q_r^\nu} |\zeta(y) - \widetilde{w}(y)|\,\mathrm{d}y.$$

We conclude

$$g(x_0,\mathrm{v}^+ - \mathrm{v}^-,\nu) \leq f^\infty\big(x_0,(\mathrm{v}^+ - \mathrm{v}^+)\odot\nu\big)$$

by letting first $\delta \downarrow 0$ and then $r \uparrow 1$ in the latter inequality. $\qquad\square$

In view of the previous corollary we are able to characterize explicitly the relaxed functional $\mathcal{F}_0$ in the v-independent case. Additionally, as a consequence, we are also able to deal with the issue of L^1 lower semicontinuity on BD. In particular, we improve upon [42] and [4, Corollary 1.10] (in the curl-curl case according to terminology used there) dispensing with the existence of the strong recession function. Note that f_0 is allowed to depend on x and that the full continuity of f_0 is not required, being replaced by the weaker assumption (H2$'$).

To state the result recall that given $f : \mathbb{M}_{sym}^{n\times n} \to \mathbb{R}$ a Borel function, its symmetric quasiconvex envelope $SQf : \mathbb{M}_{sym}^{n\times n} \to \mathbb{R}$ is defined to be

$$SQf(\mathbb{A}) := \sup\{h(\mathbb{A}) \mid h \leq f,\ \ h \text{ is symmetric quasiconvex}\}.$$

Clearly, f is symmetric quasiconvex if and only if $f = SQf$. Finally, if $f : \Omega \times \mathbb{M}_{sym}^{n\times n} \to [0,+\infty)$ we write $SQf(x,\cdot) := SQ(f(x,\cdot))$ for all $x \in \Omega$.

COROLLARY 6.8. *Let* $f_0 : \Omega \times \mathbb{M}_{sym}^{n\times n} \to [0,+\infty)$ *be a Borel function satisfying* (H1$'$)–(H2$'$). *Let* $F_0 : L^1(\Omega;\mathbb{R}^n) \times \mathcal{O}(\Omega) \to [0,+\infty]$ *be the functional defined in* (6.3) *corresponding to* f_0.

Then, for all $(u, A) \in BD(\Omega) \times \mathcal{O}(\Omega)$

$$(6.9) \quad \mathcal{F}_0(u, A) = \int_A SQf_0\big(x, e(u)(x)\big) \, \mathrm{d}x + \int_A (SQf_0)^\infty \left(x, \frac{\mathrm{d}E^s u}{\mathrm{d}|E^s u|}(x) \right) \mathrm{d}|E^s u|.$$

In particular, if $f_0 : \Omega \times \mathbb{M}^{n \times n}_{sym} \to [0, +\infty)$ *is a Borel symmetric quasiconvex satisfying* (H1′)–(H2′), *the functional* $\widetilde{\mathcal{F}}_0 : L^1(\Omega; \mathbb{R}^n) \times \mathcal{O}(\Omega) \to [0, +\infty]$ *defined by*

$$(6.10) \quad \widetilde{\mathcal{F}}_0(u, A) := \int_A f_0\big(x, e(u)(x)\big) \, \mathrm{d}x + \int_A f_0^\infty \left(x, \frac{\mathrm{d}E^s u}{\mathrm{d}|E^s u|}(x) \right) \mathrm{d}|E^s u|$$

if $u \in BD(\Omega)$ *and* $+\infty$ *otherwise, is* $L^1(\Omega; \mathbb{R}^n)$ *lower semicontinuous.*

Proof. We start off noting that by (6.4), by inequality $SQf_0 \leq f_0$, and by the symmetric quasiconvexity of SQf_0 we get

$$f(x_0, \mathbb{A}) = \inf_{\substack{w \in LD(Q_1) \\ w|_{\partial Q_1} = \mathbb{A}y|_{\partial Q_1}}} \int_Q f_0\big(x_0, e(w)(y)\big) \, \mathrm{d}y$$

$$\geq \inf_{\substack{w \in LD(Q_1) \\ w|_{\partial Q_1} = \mathbb{A}y|_{\partial Q_1}}} \int_{Q_1} SQf_0\big(x_0, e(w)(y)\big) \, \mathrm{d}y \geq SQf_0(x_0, \mathbb{A}).$$

As noticed in Corollary 6.7, the bulk energy density f is symmetric quasiconvex and $f \leq f_0$. Thus, by the very definition of SQf_0, we get $f \leq SQf_0$. Therefore, the representation for $\mathcal{F}_0$ in (6.9) is attained thanks to Corollary 6.7.

Finally, the $L^1(\Omega; \mathbb{R}^n)$ lower semicontinuity of $\widetilde{\mathcal{F}}_0$ in (6.10) follows at once. □

6.2. Relaxation of bulk and interfacial energies. In this section we consider linear functionals defined on the subspace $SBD(\Omega)$ and provide a relaxation result for them. To our knowledge this is the first result of this kind.

We introduce the notation required for our purposes following [7, section 4.2]. Let $f_1 : \Omega \times \mathbb{R}^n \times \mathbb{M}^{n \times n}_{sym} \to [0, \infty)$ and $g_1 : \Omega \times (\mathbb{R}^n)^2 \times \mathbb{S}^{n-1} \to [0, \infty)$ be continuous integrands such that

(H1″) there exists a constant $C > 0$ such that for every $(x, \mathrm{v}, \mathbb{A}) \in \Omega \times \mathbb{R}^n \times \mathbb{M}^{n \times n}_{sym}$

$$(6.11) \qquad \frac{1}{C}|\mathbb{A}| \leq f_1(x, \mathrm{v}, \mathbb{A}) \leq C\big(1 + |\mathbb{A}|\big);$$

(H2″) there exists a constant $C > 0$ such that for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$|x - x_0| + |\mathrm{v} - \mathrm{v}_0| \leq \delta \Longrightarrow |f_1(x, \mathrm{v}, \mathbb{A}) - f_1(x_0, \mathrm{v}_0, \mathbb{A})| \leq C\varepsilon(1 + |\mathbb{A}|),$$

for every $(x, x_0, \mathrm{v}, \mathrm{v}_0, \mathbb{A}) \in (\Omega)^2 \times (\mathbb{R}^n)^2 \times \mathbb{M}^{n \times n}_{sym}$;

(H3″) there exist $C > 0$ such that for every $(x, \mathrm{v}^-, \mathrm{v}^+, \nu) \in \Omega \times (\mathbb{R}^n)^2 \times \mathbb{S}^{n-1}$

$$\frac{1}{C}|\mathrm{v}^+ - \mathrm{v}^-| \leq g_1(x, \mathrm{v}^-, \mathrm{v}^+, \nu) \leq C|\mathrm{v}^+ - \mathrm{v}^-|;$$

(H4″) there exist $C > 0$, such that for every $(x, x_0, \mathrm{v}^-, \mathrm{v}^+, \mathrm{v}_0, \nu) \in (\Omega)^2 \times (\mathbb{R}^n)^3 \times \mathbb{S}^{n-1}$

$$|x - x_0| + |\mathrm{v}_0| \leq \delta \Longrightarrow |g_1(x, \mathrm{v}^- + \mathrm{v}_0, \mathrm{v}^+ + \mathrm{v}_0, \nu) - g_1(x_0, \mathrm{v}^-, \mathrm{v}^+, \nu)| \leq C\varepsilon|\mathrm{v}^+ - \mathrm{v}^-|.$$

Let then $\mathcal{F}_1 : L^1(\Omega; \mathbb{R}^n) \times \mathcal{O}(\Omega) \to [0, +\infty]$ be the functional defined by

$$(6.12) \qquad \mathcal{F}_1(u, A) := \inf \left\{ \liminf_j F_1(u_j, \Omega) : u_j \to u \text{ in } L^1(\Omega; \mathbb{R}^n) \right\},$$

namely the $L^1(\Omega; \mathbb{R}^n)$ lower semicontinuos envelope of the functional

$$(6.13)$$

$$F_1(u, A) := \begin{cases} \displaystyle\int_A f_1\big(x, u(x), e(u)(x)\big) \, \mathrm{d}x \\ \displaystyle + \int_{J_u \cap A} g_1\big(x, u^-(x), u^+(x), \nu_u(x)\big) \, \mathrm{d}\mathcal{H}^{n-1}(x) & \text{if } u \in SBD(\Omega), \\ \\ +\infty & \text{otherwise on } L^1(\Omega; \mathbb{R}^n). \end{cases}$$

Denote by $\mathfrak{m}$ the cell formula defined in (2.4) related to $\mathcal{F}_1$. We provide next an integral representation result for $\mathcal{F}_1$

THEOREM 6.9. *Let* $F_1 : L^1(\Omega; \mathbb{R}^n) \times \mathcal{O}(\Omega) \to [0, +\infty]$ *be the functional defined in* (6.13). *Then, assuming* (H1″)–(H4″), *the functional* $\mathcal{F}_1 : L^1(\Omega; \mathbb{R}^n) \times \mathcal{O}(\Omega) \to [0, +\infty]$ *defined in* (6.12) *is represented by*

$$\mathcal{F}_1(u, A) = \int_A f\big(x, u(x), e(u)(x)\big) \, \mathrm{d}x$$
$$+ \int_{J_u \cap A} g\big(x, u^-(x), u^+(x), \nu_u(x)\big) \, \mathrm{d}\mathcal{H}^{n-1}(x)$$
$$+ \int_A f^\infty\left(x, u(x), \frac{\mathrm{d}E^c u}{\mathrm{d}|E^c u|}(x)\right) \mathrm{d}|E^c u|(x)$$

for all $(u, A) \in BD(\Omega) \times \mathcal{O}(\Omega)$, *where for every* $(x_0, \mathrm{v}, \mathbb{A}) \in \Omega \times \mathbb{R}^n \times \mathbb{M}^{n \times n}_{sym}$

$$f(x_0, \mathrm{v}, \mathbb{A}) = \limsup_{\varepsilon \to 0} \inf_{\substack{w \in SBD(Q_1) \\ w|_{\partial Q_1} = \mathbb{A}y|_{\partial Q_1}}} \left\{ \int_{Q_1} f_1\big(x_0, \mathrm{v} + \varepsilon w(y), e(w)(y)\big) \, \mathrm{d}y \right.$$

$$(6.14) \qquad \left. + \int_{J_w \cap Q_1} \frac{1}{\varepsilon} g_1(x_0, \mathrm{v} + \varepsilon w^-(y), \mathrm{v} + \varepsilon w^+(y), \nu_w(y)) \, \mathrm{d}\mathcal{H}^{n-1}(y) \right\},$$

and for every $(x_0, \mathrm{v}^-, \mathrm{v}^+, \nu) \in \Omega \times (\mathbb{R}^n)^2 \times \mathbb{S}^{n-1}$

$$g(x_0, \mathrm{v}^-, \mathrm{v}^+, \nu) = \limsup_{\varepsilon \to 0} \inf_{\substack{w \in SBD(Q_1^\nu) \\ w|_{\partial Q_1^\nu} = u_{\mathrm{v}^-, \mathrm{v}^+, \nu}|_{\partial Q_1^\nu}}} \left\{ \int_{Q_1^\nu} \varepsilon f_1\big(x_0, \mathrm{v} + \varepsilon w(y), \frac{1}{\varepsilon} e(w)(y)\big) \, \mathrm{d}y \right.$$

$$(6.15) \qquad \left. + \int_{J_w \cap Q_1^\nu} g_1(x_0, w^-(y), w^+(y), \nu_w(y)) \, \mathrm{d}\mathcal{H}^{n-1}(y) \right\}.$$

Remark 6.10. In the *BV* setting under study in [7], conditions (H3″) and (H4″) are additionally used to simplify the analogue of formulas (6.14) for f and (6.15) for g thanks to the truncation argument quoted in Remark 6.2 (cf. [7, (4.2.3) and (4.2.4) in Theorem 4.2.2]).

Remark 6.11. Formulas (6.14) for f and (6.15) for g can be expressed in terms of the recession functions of f_1 at ∞, f_1^∞ and the recession function of g_1 at 0, g_1^0, provided some further technical conditions in the spirit of (H3$'$) in Corollary 6.6 are imposed (cf. [7, (4.2.3)$'$ and (4.2.4)$'$ in Remark 4.2.3]).

The proof of Theorem 6.9 is similar to the corresponding one of [7, Theorem 4.2.2]. First, we note that arguing as in Lemma 6.5 one can prove the following result.

LEMMA 6.12. *Assume* (H1$''$)–(H4$''$). *Then,* $\mathcal{F}_1$ *satisfies* (H1)–(H5), *and for all* $(u, A) \in BD(\Omega) \times \mathcal{O}_\infty(\Omega)$

$$\mathfrak{m}(u, A) = \mathfrak{m}_1(u, A) := \inf\{F_1(w, A) : w = u \text{ on } \partial A\}.$$

The alternative characterization of $\mathfrak{m}$ via $\mathfrak{m}_1$, Theorem 2.3, the results in section 4, and a change of variable provide the proof of Theorem 6.9.

We give next an explicit application of Theorem 6.9.

COROLLARY 6.13. *Let* $f_1 : \Omega \times \mathbb{M}_{sym}^{n \times n} \to [0, +\infty)$ *be a Borel, symmetric quasiconvex function satisfying* (H1$''$)–(H2$''$).

If $F_1 : L^1(\Omega; \mathbb{R}^n) \times \mathcal{O}(\Omega) \to [0, +\infty]$ *is the functional in* (6.13) *corresponding to* f_1 *and* $g_1 = f_1^\infty$, *then*

$$(6.16) \qquad \mathcal{F}_1(u, A) = \int_A f_1\big(x, e(u)(x)\big)\, \mathrm{d}x + \int_A f_1^\infty\left(x, \frac{\mathrm{d}E^s u}{\mathrm{d}|E^s u|}(x)\right) \mathrm{d}|E^s u|$$

if $u \in BD(\Omega)$ *and* $+\infty$ *otherwise.*

Proof. We start off defining

$$\widetilde{F}_1(u, A) := \begin{cases} \displaystyle\int_A f_1\big(x, e(u)(x)\big)\, \mathrm{d}x & \text{if } u \in LD(\Omega), \\ +\infty & \text{otherwise on } L^1(\Omega; \mathbb{R}^n). \end{cases}$$

Denote by $\widetilde{\mathcal{F}}_1$ the $L^1(\Omega; \mathbb{R}^n)$ lower semicontinuous envelope of $\widetilde{F}_1$. Note that $\mathcal{F}_1 \leq F_1 \leq \widetilde{F}_1$, therefore $\mathcal{F}_1 \leq \widetilde{\mathcal{F}}_1$. On the other hand, Corollary 6.8 shows that $\widetilde{\mathcal{F}}_1$ coincides with the right-hand side of (6.16), therefore $\widetilde{\mathcal{F}}_1 \leq F_1$, and then $\widetilde{\mathcal{F}}_1 \leq \mathcal{F}_1$. □

6.3. Periodic homogenization of bulk energies. In this section we briefly show how to apply Theorem 2.3 to a homogenization problem in BD. More precisely, we identify the $\Gamma(L^1(\Omega; \mathbb{R}^n))$-limit of the family of functionals $F_\delta : L^1(\Omega; \mathbb{R}^n) \times \mathcal{O}(\Omega) \to [0, +\infty]$, $\delta > 0$, given by

$$(6.17) \qquad F_\delta(u, A) := \begin{cases} \displaystyle\int_A f_0\Big(\frac{x}{\delta}, e(u)(x)\Big)\, \mathrm{d}x & \text{if } u \in LD(\Omega), \\ +\infty & \text{otherwise on } L^1(\Omega; \mathbb{R}^n), \end{cases}$$

where $f_0 : \mathbb{R}^n \times \mathbb{M}_{sym}^{n \times n} \to [0, +\infty)$ is any Borel function satisfying
(Hom) $f_0(\cdot, \mathbb{A})$ is Q_1-periodic for all $\mathbb{A} \in \mathbb{M}_{sym}^{n \times n}$ and

$$\frac{1}{C}|\mathbb{A}| \leq f_0(x, \mathbb{A}) \leq C(1 + |\mathbb{A}|)$$

for all $(x, \mathbb{A}) \in \mathbb{R}^n \times \mathbb{M}_{sym}^{n \times n}$ and for some universal constant $C > 0$.

Homogenization type problems are well-known, and the literature on the topic is very rich; we refer to the book [9] for an exhaustive introduction in particular in the case of variational energies defined on Sobolev spaces (see also [16] for more classical results), to [10] for homogenization issues related to energies defined on suitable subspaces of SBV, to [6], to [19], and to [7, section 4.3] for energies defined on BV, and to [37] for linear growth functionals in the setting of $\mathcal{A}$-quasiconvexity under additional conditions both on the differential operator $\mathcal{A}$, ruling out the curl-curl setting considered in this section, and on the density f_0, i.e., $f_0(x, \cdot)$ is Lipschitz continuous uniformly in $x \in \mathbb{R}^n$.

For energy densities $f_0(x, \cdot)$ convex for all $x \in \mathbb{R}^n$, the homogenization of the energies in (6.17) on $BD(\Omega)$ has been investigated in [6, Theorem 3.2] by duality methods. In this subsection, we extend such a result to the general case under the sole assumption (Hom) thanks to the global method for relaxation. In this perspective, Corollary 6.8 and [2, item (ii) of Lemma 4.3] will be instrumental.

In addition, let us remark that the argument used in Theorem 6.14 below works also in the setting of periodic homogenization of bulk energies defined on BV, thus recovering the results contained in [6, Theorem 3.1], for $f_0(x, \cdot)$ convex for all $x \in \mathbb{R}^n$, and in [19, Theorem 4.7], for $f_0(x, \cdot)$ continuous for $\mathcal{L}^n$-a.e. $x \in \mathbb{R}^n$, via the global method for relaxation without any additional assumption on f_0 than (Hom). Finally, we remark that [7, section 4.3] deals with the analogous problem in BV for energies consisting more generally of the sum of a volume and of a surface term. We shall not deal with that more general setting here, which will be the object of further studies. Despite this, our result for the homogenization of bulk integrals seems to be new for what the regularity of f_0 is concerned in the BV setting, as well.

We recall for the reader's convenience the definition of $\Gamma(d)$-convergence for a family of functionals $F_\delta : (X, d) \to \mathbb{R} \cup \{\pm\infty\}$ defined on a metric space (X, d): $\{F_\delta\}_{\delta>0}$ $\Gamma(d)$-converges to $\mathcal{F} : (X, d) \to \mathbb{R} \cup \{\pm\infty\}$ if for every infinitesimal sequence δ_i the ensuing two conditions are satisfied:

(i) For all $x \in X$ and for all $x_i \xrightarrow{d} x$ it holds that $\liminf_{i \to +\infty} F_{\delta_i}(x_i) \geq \mathcal{F}(x)$.

(ii) For all $x \in X$ there exists $x_i \xrightarrow{d} x$ such that $\limsup_{i \to +\infty} F_{\delta_i}(x_i) \leq \mathcal{F}(x)$.

We refer to the books [16], [8] and to the survey [25] for several properties of Γ-convergence in metric spaces.

In what follows, we use the shorthand notation $u_{v,\nu}$ for the piecewise constant function $u_{v,0,\nu}$ defined in (2.8).

We are now ready to prove the following homogenization result.

THEOREM 6.14. *Let $f_0 : \mathbb{R}^n \times \mathbb{M}_{sym}^{n \times n} \to [0, +\infty)$ be a Borel function satisfying* (Hom).

The family $F_\delta : L^1(\Omega; \mathbb{R}^n) \times \mathcal{O}(\Omega) \to [0, +\infty]$ defined in (6.17) $\Gamma(L^1(\Omega; \mathbb{R}^n))$-converges to the functional $\mathcal{F}_{\mathrm{hom}} : L^1(\Omega; \mathbb{R}^n) \times \mathcal{O}(\Omega) \to [0, +\infty]$ given for $(u, A) \in BD(\Omega) \times \mathcal{O}(\Omega)$ by

$$(6.18) \qquad \mathcal{F}_{\mathrm{hom}}(u, A) := \int_A f_{\mathrm{hom}}\big(e(u)(x)\big)\, \mathrm{d}x + \int_A f_{\mathrm{hom}}^\infty \left(\frac{\mathrm{d}E^s u}{\mathrm{d}|E^s u|}(x) \right) \mathrm{d}|E^s u|$$

and $+\infty$ otherwise on $L^1(\Omega; \mathbb{R}^n)$, where $f_{\mathrm{hom}} : \mathbb{M}_{sym}^{n \times n} \to [0, +\infty)$ is defined by

$$(6.19) \qquad f_{\mathrm{hom}}(\mathbb{A}) := \lim_{T \to +\infty} \frac{1}{T^n} \inf_{\substack{w \in LD(Q_T) \\ w|_{\partial Q_T} = \mathbb{A}y|_{\partial Q_T}}} \int_{Q_T} f_0\big(x, e(w)(x)\big)\, \mathrm{d}x \,,$$

and where f_{hom}^∞ denotes the recession function of f_{hom}.

The first part of the proof of Theorem 6.14 is a simplification of that of [7, Theorem 4.3.1] since we deal only with bulk energies (cf. Steps 1 and 2). For this reason, we give a concise proof providing precise references whenever details are omitted. Instead, the second part is based upon a homogenization type argument contained in [2, item (ii) of Lemma 4.3] (cf. Step 3).

Proof. We divide the proof into intermediate steps for the sake of convenience.

We start off recalling some abstract compactness properties that follow from the general theory of $\overline{\Gamma}$-convergence (cf. [16, Chapters 14, 16, 18]). In view of the growth condition on f_0 in (Hom), given any sequence $\delta_i \downarrow 0$, we can extract a subsequence δ_{i_j} such that the $\Gamma(L^1(\Omega; \mathbb{R}^n))$-limit of $F_{\delta_{i_j}}(\cdot, A)$ exists for all $A \in \mathcal{O}(\Omega)$ (cf. [16, Propositions 16.9, 18.6] and [7, Lemma 4.3.4]). For the sake of notational convenience we denote the subsequence δ_{i_j} simply by δ_j and by $\mathcal{F}$ the corresponding $\Gamma(L^1(\Omega; \mathbb{R}^n))$-limit.

In what follows, we shall show that $\mathcal{F}$ does not depend on the chosen subsequence, thus proving that the $\Gamma(L^1(\Omega; \mathbb{R}^n))$-limit of the family $\{F_\delta\}_{\delta>0}$ exists by Urysohn's property (cf. [16, Proposition 16.8]). To this aim, we shall first show that each functional $\mathcal{F}$ introduced above satisfies the assumptions of the integral representation Theorem 2.3, and then we shall identify its energy densities with f_{hom} for the absolutely continuous part and with f_{hom}^∞ for the singular part.

Step 1*: Integral representation of $\mathcal{F}$.* We start off noting that [10, Lemma 3.7] (see also [7, Lemma 4.3.3]) implies that $\mathcal{F}$ satisfies (H4) and (H5), namely, one can prove that

$$(6.20) \qquad \mathcal{F}(u(\cdot - x_0) + \mathbb{L} \cdot + \mathrm{v}, x_0 + A) = \mathcal{F}(u, A)$$

for all $(u, A, \mathrm{v}, x_0, \mathbb{L}) \in BD(\Omega) \times \mathcal{O}(\Omega) \times (\mathbb{R}^n)^2 \times \mathbb{M}_{skew}^{n \times n}$, with $x_0 + A \subseteq \Omega$. The very definition of $\mathcal{F}$ gives immediately (H1), (Hom) implies (H2), and finally arguing as in [7, Lemma 4.3.4] together with [5, Proposition 3.7] yields (H3).

In particular, we can apply Theorem 2.3 to deduce that $\mathcal{F}$ can be represented as

(6.21)

$$\mathcal{F}(u, A) = \int_A f\big(e(u)(x)\big)\, \mathrm{d}x$$
$$+ \int_{J_u \cap A} g\big(u^+(x) - u^-(x), \nu_u(x)\big)\, \mathrm{d}\mathcal{H}^{n-1}(x) + \int_A f^\infty\left(\frac{\mathrm{d}E^c u}{\mathrm{d}|E^c u|}(x)\right) \mathrm{d}|E^c u|(x),$$

where f and g are identified by the cell formulas (2.5) and (2.6), respectively. Indeed, on account of (6.20) with $x_0 = \underline{0}$, Remark 5.2 implies that $f(x, \mathrm{v}, \mathbb{A}) = f(x, \mathbb{A})$ and $g(x, \mathrm{v}^-, \mathrm{v}^+, \nu) = g(x, \mathrm{v}^+ - \mathrm{v}^-, \nu)$. Instead, if in (6.20) one chooses $\mathbb{L} = \underline{0}$ and $\mathrm{v} = \underline{0}$, we infer that f and g do not depend on x.

Finally, since $\mathcal{F}$ is translation invariant we may assume that $\underline{0} \in \Omega$; this will simplify the notation in what follows.

Step 2*: $f = f_{\mathrm{hom}}$.* First we claim that the limit defining $f_{\mathrm{hom}}(\mathbb{A})$ exists finite for all $\mathbb{A} \in \mathbb{M}_{sym}^{n \times n}$. Indeed, with fixed $\mathbb{A} \in \mathbb{M}_{sym}^{n \times n}$, the proof of such a claim follows by applying the global ergodic theorem in [36, Theorem 2.1] (or [36, Theorem 3.1]) to the $\mathbb{Z}^n$-invariant subadditive process $\mathcal{S} : \mathcal{O}_\infty(\mathbb{R}^n) \to [0, +\infty]$ defined by $\mathcal{S}(A) := \mathfrak{m}(\mathbb{A}x; A)$ (for more details cf. (4.3.15) in [7, Lemma 4.3.7] and [7, Lemma 4.3.6]). In particular, we note that the conclusions of [36, Theorem 2.1] are true also for subadditive processes defined only on bounded open sets with Lipschitz boundary, as outlined in the remark at the end of [36, section 2.1].

For $(u, A) \in BD(\Omega) \times \mathcal{O}_\infty(\Omega)$ and $j \in \mathbb{N}$, consider the intermediate cell problems

$$\mathfrak{m}_{\delta_j}(u, A) := \inf \left\{ F_{\delta_j}(u, A) : w \in LD(A), \quad w = u \text{ on } \partial A \right\}.$$

With fixed $x_0 \in \Omega$ using the version of the De Giorgi's averaging/slicing lemma developed in [5, Proposition 3.7] one can argue as in [7, Lemma 4.3.5] to establish the convergence

$$(6.22) \qquad \liminf_{j \to +\infty} \mathfrak{m}_{\delta_j}(u, Q(x_0, r)) = \mathfrak{m}(u, Q(x_0, r))$$

for $\mathcal{L}^1$-a.e. $r \in (0, \frac{2}{\sqrt{n}} d(x_0, \partial \Omega))$, where $\mathfrak{m}$ is the cell formula for $\mathcal{F}$ defined in (2.4).

Thus, by taking into account (2.5) and (6.22) we conclude that

$$(6.23) \qquad f(\mathbb{A}) = \lim_{\varepsilon \to 0} \liminf_{j \to +\infty} \frac{\mathfrak{m}_{\delta_j}(\mathbb{A}x, Q_\varepsilon)}{\varepsilon^n}.$$

We prove $f \geq f_{\text{hom}}$. With fixed $\mathbb{A} \in \mathbb{M}^{n \times n}_{sym}$, select δ_{j_ε}, with $\delta_{j_\varepsilon}/\varepsilon \to 0$ as $\varepsilon \downarrow 0$, and $v_\varepsilon \in LD(Q_\varepsilon)$, with $v_\varepsilon|_{\partial Q_\varepsilon} = \mathbb{A}x|_{\partial Q_\varepsilon}$, such that

$$f(\mathbb{A}) = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon^n} F_{\delta_{j_\varepsilon}}(v_\varepsilon, Q_\varepsilon).$$

Set $T_\varepsilon := \varepsilon/\delta_{j_\varepsilon} \uparrow +\infty$ and $w_\varepsilon(y) := \frac{1}{\delta_{j_\varepsilon}} v_\varepsilon(\delta_{j_\varepsilon} y) \in LD(Q_{T_\varepsilon})$; then $w_\varepsilon|_{\partial Q_{T_\varepsilon}} = \mathbb{A}x|_{\partial Q_{T_\varepsilon}}$ and by a change of variables the very definition of $f_{\text{hom}}(\mathbb{A})$ yields

$$f(\mathbb{A}) = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon^n} F_{\delta_{j_\varepsilon}}(v_\varepsilon, Q_\varepsilon) = \lim_{\varepsilon \to 0} \frac{1}{T_\varepsilon^n} \int_{Q_{T_\varepsilon}} f_0\big(z, e(w_\varepsilon)(z)\big) \, \mathrm{d}z \geq f_{\text{hom}}(\mathbb{A}).$$

For the converse inequality $f \leq f_{\text{hom}}$, for every $T > 0$ choose $u_T \in LD(Q_T)$ such that $u_T|_{\partial Q_T} = \mathbb{A}x|_{\partial Q_T}$ and

$$f_{\text{hom}}(\mathbb{A}) = \lim_{T \to +\infty} \frac{1}{T^n} \int_{Q_T} f_0\big(x, e(u_T)(x)\big) \, \mathrm{d}x.$$

For every $\varepsilon > 0$ and $j \in \mathbb{N}$, let $T_{\varepsilon,j} = \varepsilon/\delta_j > 0$, and set $w_{\varepsilon,j}(z) := \delta_j u_{T_{\varepsilon,j}}\big(\frac{z}{\delta_j}\big)$. For every $\varepsilon > 0$ note that $T_{\varepsilon,j} \uparrow +\infty$ as $j \uparrow +\infty$, $w_{\varepsilon,j}|_{\partial Q_\varepsilon} = \mathbb{A}z|_{\partial Q_\varepsilon}$ and thus by changing variables that

$$f_{\text{hom}}(\mathbb{A}) = \lim_{j \to +\infty} \frac{1}{T_{\varepsilon,j}^n} \int_{Q_{T_{\varepsilon,j}}} f_0\big(z, e(u_{T_{\varepsilon,j}})(z)\big) \, \mathrm{d}z$$

$$= \lim_{j \to +\infty} \frac{1}{\varepsilon^n} \int_{Q_\varepsilon} f_0\left(\frac{z}{\delta_j}, e(w_{\varepsilon,j})(z)\right) \, \mathrm{d}z \geq \liminf_{j \to +\infty} \frac{\mathfrak{m}_{\delta_j}(\mathbb{A}z, Q_\varepsilon)}{\varepsilon^n}.$$

Therefore, (6.23) provides the inequality $f(\mathbb{A}) \leq f_{\text{hom}}(\mathbb{A})$ by letting $\varepsilon \downarrow 0$ in the inequality above.

In particular, since the argument above is independent from the chosen subsequence δ_j, we conclude that for all $(u, A) \in LD(\Omega) \times \mathcal{O}(\Omega)$

$$(6.24) \qquad \mathcal{F}(u, A) = \int_A f_{\text{hom}}\big(e(u)(x)\big) \, \mathrm{d}x$$

for all functionals $\mathcal{F}$ arising as $\Gamma(L^1(\Omega;\mathbb{R}^n))$-limits of subsequences of the family $\{F_\delta\}_{\delta>0}$. Therefore, by (6.24) the Γ-liminf $\mathcal{F}^-$ of the family $\{F_\delta\}_{\delta>0}$, defined as

$$\mathcal{F}^-(u,A) := \inf\left\{\liminf_{j\to+\infty} F_{\delta_j}(u_j,A) : u_j \to u \quad L^1(\Omega;\mathbb{R}^n), \quad \delta_j \to 0\right\}$$

satisfies

(6.25)
$$\mathcal{F}^-(u,A) = \int_A f_{\text{hom}}\big(e(u)(x)\big)\,\mathrm{d}x$$

for all $(u,A) \in LD(\Omega) \times \mathcal{O}(\Omega)$. In particular, by definition $\mathcal{F}^-(\cdot,A)$ is $L^1(\Omega;\mathbb{R}^n)$ lower semicontinuous and

(6.26)
$$\mathcal{F}(u,A) \geq \mathcal{F}^-(u,A)$$

for all $(u,A) \in LD(\Omega) \times \mathcal{O}(\Omega)$, where $\mathcal{F}$ is any $\Gamma(L^1(\Omega;\mathbb{R}^n))$-limit of a subsequence F_{δ_j}.

Step 3: $g = f_{\text{hom}}^\infty$. We start off proving the inequality $g \leq f_{\text{hom}}^\infty$. For all $(u,A) \in L^1(\Omega;\mathbb{R}^n) \times \mathcal{O}(\Omega)$ set

$$F(u,A) := \begin{cases} \displaystyle\int_A f_{\text{hom}}\big(e(u)(x)\big)\,\mathrm{d}x & \text{if } u \in LD(\Omega), \\ +\infty & \text{otherwise on } L^1(\Omega;\mathbb{R}^n); \end{cases}$$

then by Steps 1 and 2 it follows that $\mathcal{F}(u,A) \leq F(u,A)$. Denoting by $\widetilde{\mathcal{F}}$ the $L^1(\Omega;\mathbb{R}^n)$ lower semicontinuous envelope of F, then $\mathcal{F}(u,A) \leq \widetilde{\mathcal{F}}(u,A)$. Hence, we conclude $g \leq f_{\text{hom}}^\infty$ in view of Corollary 6.8 that provides the explicit expression of $\widetilde{\mathcal{F}}$.

To prove $g \geq f_{\text{hom}}^\infty$ we argue as follows. For every $\varepsilon > 0$, we introduce the family of auxiliary functionals $F_{\delta_j}^{(\varepsilon)}$, $\mathcal{F}^{(\varepsilon)} : L^1(\frac{1}{\varepsilon}\Omega;\mathbb{R}^n) \times \mathcal{O}(\frac{1}{\varepsilon}\Omega) \to [0,+\infty]$ defined by

$$F_{\delta_j}^{(\varepsilon)}(u,A) := \begin{cases} \displaystyle\int_A f_0\Big(\frac{\varepsilon}{\delta_j}x, e(u)(x)\Big)\,\mathrm{d}x & \text{if } u \in LD(\tfrac{1}{\varepsilon}\Omega), \\ +\infty & \text{otherwise on } L^1(\tfrac{1}{\varepsilon}\Omega;\mathbb{R}^n), \end{cases}$$

and

$$\mathcal{F}^{(\varepsilon)}(w,A) := \int_A f_{\text{hom}}\big(e(w)(x)\big)\,\mathrm{d}x$$
$$+ \int_{J_w \cap A} \frac{1}{\varepsilon}g\big(\varepsilon(w^+(x) - w^-(x)), \nu_w(x)\big)\,\mathrm{d}\mathcal{H}^{n-1}(x) + \int_A f_{\text{hom}}^\infty\left(\frac{\mathrm{d}E^c w}{\mathrm{d}|E^c w|}(x)\right)\mathrm{d}|E^c w|.$$

For $w \in BD(\Omega)$ set $w_\varepsilon(y) := \frac{1}{\varepsilon}w(\varepsilon y) \in BD(\frac{1}{\varepsilon}\Omega)$; then a simple scaling argument yields that (cf. Remark 3.7)

$$F_{\delta_j}(w,A) = \varepsilon^n F_{\delta_j}^{(\varepsilon)}\left(w_\varepsilon, \frac{1}{\varepsilon}A\right) \quad \text{and} \quad \mathcal{F}(w,A) = \varepsilon^n \mathcal{F}^{(\varepsilon)}\left(w_\varepsilon, \frac{1}{\varepsilon}A\right),$$

so that $\mathcal{F}^{(\varepsilon)}(u,O) = \Gamma(L^1(\frac{1}{\varepsilon}\Omega;\mathbb{R}^n))\text{-}\lim_{j\to+\infty} F_{\delta_j}^{(\varepsilon)}(u,O)$ for all $(u,O) \in BD(\frac{1}{\varepsilon}\Omega) \times \mathcal{O}(\frac{1}{\varepsilon}\Omega)$. In addition, for all $(\mathrm{v},\nu) \in \mathbb{R}^n \times \mathbb{S}^{n-1}$, by (2.6) we deduce that

$$g(\mathrm{v},\nu) = \lim_{\varepsilon\to 0} \frac{\mathfrak{m}(u_{\mathrm{v},\nu}, Q_\varepsilon^\nu)}{\varepsilon^{n-1}} = \lim_{\varepsilon\to 0} \varepsilon\inf\left\{\mathcal{F}^{(\varepsilon)}(w, Q_1^\nu) : w = \frac{1}{\varepsilon}u_{\mathrm{v},\nu} \text{ on } \partial Q_1^\nu\right\}.$$

We next claim that for all $\varepsilon > 0$

$$(6.27) \qquad \inf \left\{ \mathcal{F}^{(\varepsilon)}(w, Q_1^\nu) : \ w = \frac{1}{\varepsilon} u_{\mathrm{v}, \nu} \text{ on } \partial Q_1^\nu \right\} \geq f_{\mathrm{hom}} \left(\frac{\mathrm{v} \odot \nu}{\varepsilon} \right).$$

Given this for granted, we infer that

$$g(\mathrm{v}, \nu) \geq \limsup_{\varepsilon \to 0} \varepsilon \, f_{\mathrm{hom}} \left(\frac{\mathrm{v} \odot \nu}{\varepsilon} \right) = f_{\mathrm{hom}}^\infty (\mathrm{v} \odot \nu).$$

Thus, to conclude we are left with the proof of (6.27). To this aim we use a construction introduced in [2, item (ii) of Lemma 4.3] and adapt their argument to our setting. By scaling we assume that $Q_2 \subset \Omega$, and for the sake of simplicity we assume $\nu = \mathrm{e}_1$. In particular, $\mathcal{F}^{(\varepsilon)}(w, Q_1) = \mathcal{F}^{(\varepsilon)}(w(\cdot + \mathrm{e}), (0,1)^n)$ for every $\varepsilon > 0$ and $w \in BD(Q_1)$, where $\mathrm{e} := \frac{1}{2}(\mathrm{e}_1 + \cdots + \mathrm{e}_n)$. Therefore, in place of (6.27) we shall equivalently prove that for all $\varepsilon > 0$

$$(6.28) \qquad \inf \left\{ \mathcal{F}^{(\varepsilon)}(w, (0,1)^n) : \ w = \frac{1}{\varepsilon} u_{\mathrm{v}, \nu}(\cdot - \mathrm{e}) \text{ on } \partial (0,1)^n \right\} \geq f_{\mathrm{hom}} \left(\frac{\mathrm{v} \odot \nu}{\varepsilon} \right).$$

We introduce next some notation. $\lfloor t \rfloor$ stands for the integer part of $t \in \mathbb{R}$, and with a slight abuse set $\lfloor x \rfloor := (\lfloor x_1 \rfloor, \ldots, \lfloor x_n \rfloor)$ for all $x \in \mathbb{R}^n$. Then, for every $w \in BD((0,1)^n)$ with $w = \frac{1}{\varepsilon} u_{\mathrm{v}, \nu}(\cdot - \mathrm{e})$ on $\partial (0,1)^n$, set

$$w_j(x) := \frac{1}{j} \left(w \left(jx - \lfloor jx \rfloor \right) + \frac{1}{\varepsilon} \lfloor jx_1 \rfloor \mathrm{v} \right)$$

for all $j \in \mathbb{N}$ and for all $x \in (0,1)^n$. The sequence $\{w_j\}_{j \in \mathbb{N}}$ is bounded in $BD((0,1)^n)$ and converges to the affine function $w_\infty(x) := \frac{x_1}{\varepsilon} \mathrm{v}$ strongly in $L^1((0,1)^n; \mathbb{R}^n)$ (we do not highlight the dependence of w_j and w_∞ on ε for notational convenience). Indeed, it is easy to check that $\frac{1}{j} \lfloor jx_1 \rfloor$ converges to x_1 strongly in $L^\infty((0,1)^n; \mathbb{R}^n)$ and that by Q_1-periodicity of w

$$\frac{1}{j} \int_{(0,1)^n} \left| w \left(jx - \lfloor jx \rfloor \right) \right| \mathrm{d}x = \frac{1}{j^{n+1}} \int_{(0,j)^n} |w(y - \lfloor y \rfloor)| \, \mathrm{d}y = \frac{1}{j} \int_{(0,1)^n} |w(y)| \, \mathrm{d}y.$$

Let $Q^{(1)}, \ldots, Q^{(j^n)}$ be the standard decomposition of $(0,1)^n$ into j^n congruent subcubes of side $1/j$ with sides parallel to the coordinate hyperplanes. Since by construction $|Ew_1|(\pi) = 0$ for every coordinate hyperplane of the form $\pi = \{x \in \mathbb{R}^n : x_i = c \in \mathbb{Z}^n\}$, for some $i \in \{1, \ldots, n\}$, and $w_j(x) = \frac{1}{j} w_1(jx)$, then $|Ew_j|((0,1)^n \cap \partial Q^{(k)}) = 0$, for all $k \in \{1, \ldots, j^n\}$. Then, $|Ew_j|((0,1)^n) = |Ew|((0,1)^n)$.

By translation invariance of $\mathcal{F}^{(\varepsilon j)}$ and by $(0, 1/j)^n$-periodicity of w_j we have that

$$(6.29) \qquad \mathcal{F}^{(\varepsilon j)}(w_j, (0, 1/j)^n) = \mathcal{F}^{(\varepsilon j)}(w_j, Q^{(i)})$$

for all $i \in \{1, \ldots, j^n\}$. Recalling that $f_{\mathrm{hom}}(\mathbb{A}) \leq C(1 + |\mathbb{A}|)$ for all $\mathbb{A} \in \mathbb{M}_{sym}^{n \times n}$, then $\mathcal{F}^{(\varepsilon j)}(u, A) \leq C(\mathcal{L}^n(A) + |Eu|(A))$ for all $(u, A) \in BD((0,1)^n) \times \mathcal{O}((0,1)^n)$. Therefore, $\mathcal{F}^{(\varepsilon j)}(w_j, (0,1)^n \cap \partial Q^{(k)}) = 0$, and thus from (6.29) we get that

$$(6.30) \qquad \mathcal{F}^{(\varepsilon j)}(w_j, (0,1)^n) = j^n \mathcal{F}^{(\varepsilon j)}(w_j, (0, 1/j)^n).$$

On the other hand, by changing variables $(x = y/j)$ we get that

$$(6.31) \qquad \mathcal{F}^{(\varepsilon j)}(w_j, (0, 1/j)^n) = j^{-n} \mathcal{F}^{(\varepsilon)}(w, (0,1)^n).$$

In conclusion, (6.30) and (6.31) yield that for all $\varepsilon > 0$ and for all $j \in \mathbb{N}$

$$(6.32) \qquad \mathcal{F}^{(\varepsilon)}(w, (0,1)^n) = \mathcal{F}^{(\varepsilon j)}(w_j, (0,1)^n).$$

Since $F_{\delta_j}^{(\varepsilon)}(\cdot, (0,1)^n) = F_{\delta_{j/\varepsilon}}(\cdot, (0,1)^n)$ for all $j \in \mathbb{N}$ and $\varepsilon > 0$, then $\mathcal{F}^{(\varepsilon)}$ is a $\Gamma(L^1(\Omega; \mathbb{R}^n))$-limit of a (suitable) subsequence of the family $\{F_\delta\}_{\delta > 0}$. Thus, by the $L^1(\Omega; \mathbb{R}^n)$ lower semicontinuity of the Γ-liminf functional $\mathcal{F}^-(\cdot, (0,1)^n)$, we conclude that

$$\mathcal{F}^{(\varepsilon)}(w, (0,1)^n) \overset{(6.32)}{=} \liminf_{j \to +\infty} \mathcal{F}^{(\varepsilon j)}(w_j, (0,1)^n)$$

$$\overset{(6.26)}{\geq} \liminf_{j \to +\infty} \mathcal{F}^-(w_j, (0,1)^n) \geq \mathcal{F}^-(w_\infty, (0,1)^n) \overset{(6.25)}{=} f_{\text{hom}}\left(\frac{\mathrm{v} \odot \mathrm{e}_1}{\varepsilon} \right).$$

In turn, from this (6.28) follows at once. $\qquad \square$

Remark 6.15. In the convex case it is well-known that the homogenized bulk energy density f_{hom} can be alternatively expressed for all $\mathbb{A} \in \mathbb{M}_{sym}^{n \times n}$ as

$$f_{\text{hom}}(\mathbb{A}) = \inf_{\substack{w \in LD_{loc}(\mathbb{R}^n) \\ w \ Q_1\text{-periodic}}} \int_{Q_1} f_0(x, \mathbb{A} + e(w)(x)) \, \mathrm{d}x$$

(see [9, Theorems 14.5, 14.7 and Remark 14.6] and [6, Theorem 3.1]).

7. Comments on the assumption of invariance under superposed rigid body motion. In this section we comment on the need of assumption (H5) in the BD setting. As noticed in formula (5.2), assumption (H5) implies that the bulk energy density f of $\mathcal{F}$, and then in turn its recession function f^∞, does not depend on the skew-symmetric part of the relevant matrix. This piece of information has been substantially used in the proof of Theorem 2.3 to give a lower bound of the Radon–Nikodým derivative of $\mathcal{F}$ at $u \in BD(\Omega)$ with respect to $|E^c u|$ (cf. (5.15)), the upper bound instead being always true. As far as we have understood, this seems not to be a mere technical issue as we try to explain in what follows. First we notice that there exist one-homogeneous, nonconvex, quasiconvex functions f satisfying for all $\mathbb{A} \in \mathbb{M}^{n \times n}$

$$(7.1) \qquad \frac{1}{C} \frac{|\mathbb{A} + \mathbb{A}^t|}{2} \leq f(\mathbb{A}) \leq C \frac{|\mathbb{A} + \mathbb{A}^t|}{2}$$

and depending nontrivially on the skew-symmetric part of the relevant matrix $\mathbb{A}$. The example is obtained by a slight modification of the one-homogeneous, nonconvex, quasiconvex function exhibited by Müller [39, Theorem 1] that we briefly recall. Given a matrix $\mathbb{A} \in \mathbb{M}^{2 \times 2}$ consider

$$h(\mathbb{A}) := |\mathbb{A}_{11} - \mathbb{A}_{22}| + |\mathbb{A}_{12} + \mathbb{A}_{21}| + \min\{|\mathbb{A}_{11} + \mathbb{A}_{22}|, |\mathbb{A}_{12} - \mathbb{A}_{21}|\}$$

and set Qh to be the quasiconvexification of h, i.e.,

$$Qh(\mathbb{A}) := \sup\{h'(\mathbb{A}) \mid h' \leq h, \ h' \text{ is quasiconvex}\}.$$

Clearly, Qh is quasiconvex by definition and it is easy to see that it is one-homogeneous and satisfies $0 \leq Qh(\mathbb{A}) \leq C \frac{|\mathbb{A} + \mathbb{A}^t|}{2}$ for all $\mathbb{A} \in \mathbb{M}^{2 \times 2}$. To prove the nonconvexity of Qh, Müller shows that for the rank-2 matrix

$$\mathbb{A}_0 := \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$$

$Qh(\mathbb{A}_0) > 0$ is satisfied. On the other hand, it is easy to show that the convex envelope of h is null on $\mathbb{A}_0$. In addition, to prove that Qh depends on the skew-symmetric part of the matrix $\mathbb{A}$ it suffices to notice that

$$\frac{\mathbb{A}_0 + \mathbb{A}_0^t}{2} = \mathrm{Id}, \qquad \frac{\mathbb{A}_0 - \mathbb{A}_0^t}{2} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

and that $h(\mathrm{Id}) = 0$, in turn implying $Qh(\mathrm{Id}) = 0$. Finally, in order to satisfy the growth condition in (7.1) we define the function f to be equal to

$$Qh(\mathbb{A}) + \varepsilon \frac{|\mathbb{A} + \mathbb{A}^t|}{2}$$

for $\varepsilon > 0$ sufficiently small. Indeed, the set

$$I := \left\{ \varepsilon \geq 0 : \ \mathbb{M}^{n \times n} \ni \mathbb{A} \mapsto Qh(\mathbb{A}) + \varepsilon \frac{|\mathbb{A} + \mathbb{A}^t|}{2} \text{ is convex} \right\}$$

is closed (and potentially empty), since convexity is stable under pointwise convergence, and $0 \in [0, +\infty) \setminus I$. In passing, we recall that an example of similar nature has been exhibited in the superlinear case in [14, Remark 4.14] with a polyconvex, nonconvex energy density.

Hence, the full integral representation result for the corresponding functional

$$\mathcal{F}(u, A) := \inf \left\{ \liminf_j \int_A f\big(\nabla u_j(x)\big) \, dx : u_j \to u \text{ in } L^1(\Omega; \mathbb{R}^2), \ u_j \in LD(\Omega) \right\}$$

can be proven by means neither of Theorem 2.3, since assumption (H5) is violated (while (H1)–(H4) are easily checked to be valid) nor of any of the results available in the related literature (cf. [5], [24], [42], [4], [35]). On the other hand, thanks to Lemma 5.1 (cf. [24, Theorem 3.3, Remark 3.5]), the quasiconvexity and one-homogeneity of f itself imply that for all $(u, A) \in SBD(\Omega) \times \mathcal{O}(\Omega)$ it holds that

$$(7.2) \qquad \mathcal{F}(u, A) = \int_A f\big(\nabla u(x)\big) \, dx + \int_{J_u \cap A} f\big((u^+(x) - u^-(x)) \otimes \nu_u(x)\big) \, d\mathcal{H}^1$$

(see also Theorem 6.1, Remark 6.3, and Corollary 6.8). More generally, such representations of the volume and surface parts of the energy hold for all $u \in BD(\Omega)$ in view of Lemma 5.1 itself.

In addition, for all $(u, A) \in BV(\Omega; \mathbb{R}^2) \times \mathcal{O}(\Omega)$ we claim that

$$(7.3) \qquad \mathcal{F}(u, A) = \int_A f\big(\nabla u(x)\big) \, dx + \int_{J_u \cap A} f\big((u^+(x) - u^-(x)) \otimes \nu_u(x)\big) \, d\mathcal{H}^1$$
$$+ \int_A f\left(\frac{dD^c u}{d|D^c u|}(x) \right) d|D^c u| \, .$$

To prove this, for $\delta \in (0, \varepsilon)$ consider the functional

$$\mathcal{F}_\delta(u, A) := \inf \left\{ \liminf_{j \to +\infty} \int_A f_\delta(\nabla u_j) \, dx : u_j \to u \text{ in } L^1(\Omega; \mathbb{R}^2), \ u_j \in W^{1,1}(\Omega; \mathbb{R}^2) \right\},$$

where $\mathbb{M}^{2 \times 2} \ni \mathbb{A} \mapsto f_\delta(\mathbb{A}) := f(\mathbb{A}) + \delta \frac{|\mathbb{A} - \mathbb{A}^t|}{2}$, and note that it satisfies the assumptions of the integral representation result [7, Theorem 4.1.4]. To this aim, set

$$F_\delta(u, A) := \int_A f_\delta(\nabla u) \, dx \quad \text{and} \quad F(u, A) := \int_A f(\nabla u) \, dx$$

if $u \in W^{1,1}(\Omega; \mathbb{R}^2)$ and $+\infty$ otherwise on $L^1(\Omega; \mathbb{R}^2)$. In particular, $\mathcal{F}$ turns out to be the L^1 lower semicontinuous envelope of F (cf. Remark 6.4). Furthermore, by its very definition f_δ is quasiconvex and one-homogeneous and $\delta|\mathbb{A}| \leq f_\delta(\mathbb{A}) \leq C|\mathbb{A}|$ for all $\mathbb{A} \in \mathbb{M}^{2\times 2}$, for some C independent from δ. Therefore, [7, Theorem 4.1.4] gives that for all $(u, A) \in BV(\Omega; \mathbb{R}^2) \times \mathcal{O}(\Omega)$

$$\mathcal{F}_\delta(u, A) = \int_A f_\delta(\nabla u(x)) \, \mathrm{d}x + \int_{J_u \cap A} f_\delta((u^+(x) - u^-(x)) \otimes \nu_u(x)) \, \mathrm{d}\mathcal{H}^1$$
$$+ \int_A f_\delta\left(\frac{\mathrm{d}D^c u}{\mathrm{d}|D^c u|}(x)\right) \mathrm{d}|D^c u|,$$

and moreover $\mathcal{F}_\delta(u, A) = +\infty$ if $u \in L^1 \setminus BV(\Omega; \mathbb{R}^2)$.

Being $\delta \mapsto F_\delta$ monotone increasing and converging to F as $\delta \downarrow 0$ in view of the pointwise convergence of f_δ to f, the Γ-limit in the strong $L^1(\Omega; \mathbb{R}^2)$ topology of both $(F_\delta)_\delta$ and $(\mathcal{F}_\delta)_\delta$ is exactly $\mathcal{F}$ (cf. [16, Propositions 5.7 and 6.11], [8, Theorem 1.17 and Remark 1.40]; see also [25, Example 2.5 and Theorem 2.8]). Moreover, $\delta \mapsto \mathcal{F}_\delta$ is monotone increasing with pointwise limit as $\delta \downarrow 0$ the functional $\widetilde{\mathcal{F}}$ given by the right-hand side of (7.3) for all $(u, A) \in BV(\Omega; \mathbb{R}^2) \times \mathcal{O}(\Omega)$ and $+\infty$ otherwise on $L^1(\Omega; \mathbb{R}^2)$. The equality in (7.3) then follows at once in view of the $L^1(A; \mathbb{R}^2)$ lower semicontinuity of $\widetilde{\mathcal{F}}(\cdot, A)$ itself on $BV(A; \mathbb{R}^2)$ for all $A \in \mathcal{O}(\Omega)$ (cf. [2, Theorem 4.1], [3, Theorem 5.47]). In addition, the functional $\widetilde{\mathcal{F}}$ turns out to provide the $L^1_{loc}(\Omega; \mathbb{R}^2)$ lower semicontinuous envelope of F on $(u, A) \in BV(\Omega; \mathbb{R}^2) \times \mathcal{O}(\Omega)$ in view of [2, Theorem 4.1].

Actually, as a byproduct, it follows that for all $(u, A) \in BD(\Omega) \times \mathcal{O}(\Omega)$

$$(7.4) \quad \mathcal{F}(u, A) = \inf\left\{\liminf_{j \to +\infty} \widetilde{\mathcal{F}}(u_j, A) : u_j \to u \text{ in } L^1(\Omega; \mathbb{R}^2), u_j \in BV(\Omega; \mathbb{R}^2)\right\}.$$

Summarizing, for the functional $\mathcal{F}$ related to the integrand f introduced above, we can prove an integral representation result on maps $u \in BV(\Omega; \mathbb{R}^2)$ (see (7.3)) with volume density expressed in terms of the full approximate gradient of u and with surface density depending on the full tensor product of the jump and the approximate normal to the jump set of u (cf. (7.2) and (7.3)). For $BD(\Omega)$ maps which are not $BV(\Omega; \mathbb{R}^2)$ we are not able to provide the integral representation of the Cantor term. The expression of $\mathcal{F}$ for BV maps in (7.3) suggests that despite the recent progresses obtained by De Philippis and Rindler in [21], some other structure properties of the Cantor part of the symmetrized distributional derivative are still missing for BD maps (see [20, Conjecture 3.4] for further discussions on the topic). In Theorem 2.3 we use assumption (H5) to rule out such difficulties.

As already noticed, (H5) is not needed for the integral representation on the subspace of functions $SBD(\Omega)$. The latter comment is coherent with the needed assumptions in case of functionals defined on $SBD^p(\Omega)$, $p > 1$ (see [14, Theorem 1.1 and Remark 4.14]).

Finally, we note that all the (few) known examples of functions u in $BD(\Omega) \setminus BV(\Omega; \mathbb{R}^n)$ are such that $\nabla u \notin L^1(\Omega; \mathbb{M}^{n\times n})$, though, with a slight abuse of notation, $D^s u \in \mathcal{M}(\Omega; \mathbb{M}^{n\times n})$ (cf. [41], [1, Example 7.7], [12, Theorem 1], [31], [32, Theorems 1.3 and 5.1], [15, Theorems 3.1 and 3.6]).

Remark 7.1. More generally, considering a generic functional $\mathcal{F}$ satisfying (H1)–(H4), conclusions similar to those discussed above can be drawn for what concerns the integral representation on the subspace $BV(\Omega; \mathbb{R}^n)$ and the analogous of the relaxation formula (7.4).

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