SIAM J. MATH. ANAL.
Vol. 48, No. 1, pp. 320–348


# ANALYSIS OF THE INCOMPATIBILITY OPERATOR AND APPLICATION IN INTRINSIC ELASTICITY WITH DISLOCATIONS*

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**Abstract.** The incompatibility operator arises in the modeling of elastic materials with dislocations and in the intrinsic approach to elasticity, where it is related to the Riemannian curvature of the elastic metric. It consists of applying successively the curl to the rows and the columns of a second-rank tensor, usually chosen symmetric and divergence-free. This paper presents a systematic analysis of boundary value problems associated with the incompatibility operator. It provides answers to such questions as existence and uniqueness of solutions, boundary trace lifting, and transmission conditions. Physical interpretations in dislocation models are also discussed, but the application range of these results far exceed any specific physical model.

**Key words.** incompatibility, dislocations, intrinsic elasticity, Sobolev spaces, lifting, transmission conditions

**AMS subject classifications.** 35J48, 35J58, 53A05, 74A45, 74B99

**DOI.** 10.1137/15M1020113

## 1. Introduction.

The incompatibility operator is a second-order differential operator consisting of taking the curl of the rows and the columns of a second-rank tensor $\epsilon$, viz.,

$$\mathrm{inc}\ \epsilon = \mathrm{Curl}\ \mathrm{Curl}^{\mathrm{t}}\ \epsilon, \tag{1.1}$$

the curl being taken rowwise. The incompatibility operator arises in physics, in the area of dislocation modeling, since the linear elastic strain $\epsilon$ is incompatible in the presence of dislocations, that is, cannot be written as a symmetric gradient, as soon as $\mathrm{inc}\ \epsilon \neq 0$. Specifically, its incompatibility is related to the tensor-valued density of dislocations $\Lambda$ as found by Kröner [14] and further discussed in, e.g., [20, 23], and shows ultimately as a macroscopic manifestation of plasticity (let us recall that plasticity is generated by dislocation motion). The insight of Kröner was to understand the incompatibility as a genuine geometric property of the dislocated crystal related to the connection torsion and contortion (we refer the reader to [9, 18, 20]); the crystallographic evidence of the latter had been first identified by Nye [17]. In a recent contribution [21] to this discussion, it was shown that the incompatible strain is written by virtue of the Beltrami decomposition [15] as

$$\epsilon = \nabla^S u + \mathrm{inc}\ F, \tag{1.2}$$

where $u$ may be given the meaning of a displacement field, here complemented with a tensor-valued symmetric and divergence-free field $F$ which is related to the dislocation density by the formula

$$\mathrm{inc}\ \mathrm{inc}\ F = \mathrm{inc}\ \epsilon = \mathrm{Curl}\ \kappa, \tag{1.3}$$

*Received by the editors May 6, 2015; accepted for publication (in revised form) November 3, 2015; published electronically January 26, 2016.
http://www.siam.org/journals/sima/48-1/M102011.html
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where the last equality is due to Kröner [14], and with the contortion tensor

$$\kappa := \Lambda - \frac{\mathbb{I}}{2} \operatorname{tr} \Lambda. \tag{1.4}$$

Here $\Lambda$ is the macroscopic counterpart of the mesoscopic dislocation density tensor $\Lambda_{\mathcal{L}} := \tau \otimes b\mathcal{H}^1_{\llcorner\mathcal{L}}$, where $\tau$ is the unit tangent vector to the dislocation line $\mathcal{L}$, and $b$ is its Burgers vector.

Moreover, (1.1) shows tensor $\operatorname{Curl}^{\mathrm{t}} \epsilon$, called the *Frank tensor* [23], from which the infinitesimal rotations and the displacement field are classically defined in linear elasticity. In fact, in the presence of dislocation lines, the displacement and rotation jumps around the lines are explicitly given [15] by means of recursive line integration of linear combinations of the elastic strain and Frank tensors. Obviously, these jumps vanish if and only if the strain incompatibility vanishes.

So far, we can say that incompatibility, an important operator in continuum mechanics, is related to dislocations and carries a clear geometric interpretation. About this latter point, it should be emphasized that $\operatorname{inc} \epsilon$ is another way of writing the curvature tensor associated to the elastic metric $g := \mathbb{I} - 2\epsilon$, to the first order (the explicit relation between the fourth-rank curvature tensor and the second-rank strain incompatibility can be found in [15]), whose properties in mathematical models of elasticity have been discussed in a series of recent works by Ciarlet (see, for instance, [7, 6]), in which he refers to the intrinsic model. Let us emphasize here the deep impact of this point of view for modeling, since it consists in a change of paradigm: to no longer consider the displacement as the main model variable, but rather the strain, and from this knowledge (be it given, or be it deduced from a constitutive law from the stress field) to define an elastic metric $g$, the associated Riemann curvature tensor (and possibly, the Cartan torsion, as far as dislocations are concerned; see, e.g., [14, 20]), and then, in a second step, to deduce a displacement field. In Ciarlet's series of works, the displacement is found as soon as the curvature tensor associated to $g$ vanishes. We also share this point of view and consider the elastic strain as the primal variable, defined here from the stress $\sigma$ by the linear homogeneous, isotropic, and isothermal constitutive law

$$\epsilon = \mathbb{A}^{-1}\sigma,$$

where $\mathbb{A}$ is the elasticity tensor. However, for us, not only does the Riemann curvature not vanish because of (1.3), but it is the central model variable besides the elastic strain.

The main motivation and purpose of this work is indeed the mathematical study of the incompatibility operator, in particular, in terms of function spaces used and their properties. From a mathematical viewpoint, it should also be stressed that the incompatibility operator is related to the Laplacian in the sense that for symmetric and divergence-free fields $E$, one has $\operatorname{tr} \operatorname{inc} E = \Delta \operatorname{tr} E$[1] and $\operatorname{inc} \operatorname{inc} E = \Delta^2$. Thus, in some sense, it constitutes a tensor generalization of the Laplacian, but as for the $\operatorname{Curl} \operatorname{Curl}$ operator (recall that $\operatorname{Curl} \operatorname{Curl} = -\Delta$ for solenoidal vectors), any associated boundary value problem must be addressed carefully since it applies to solenoidal fields and hence might not be an elliptic operator (see [13] for the analysis of $\operatorname{Curl} \operatorname{Curl}$ systems, the vector and first-order counterpart of our study). Indeed, the solution must satisfy the divergence-free condition in the domain, as well as specific boundary conditions (i.e., complementary in the sense of Agmon, Douglis, and

[1] Here, trace $\operatorname{tr}$ means the sum of the diagonal components of a second-rank tensor.

Nirenberg [1]), which are normal and/or tangential components of its boundary traces. Note that in $H^1$-spaces, the study of divergence-free fields is of the utmost importance in fluid dynamics [10, 19]. In particular, boundary lifting results can be found in [11].

A natural initial question is to seek the appropriate boundary conditions (if one thinks of the strong form) in order to well define the boundary value problem $\operatorname{inc}(\mathbb{M}\operatorname{inc} E) = \mathbb{G}$, or the appropriate function space (if one thinks of the weak form) to have existence of minimizers for $\int_\Omega(\frac{1}{2}\mathbb{M}\operatorname{inc} E \cdot \operatorname{inc} E - \mathbb{G}\cdot E)dx$, where $\mathbb{G}$ is some given symmetric and divergence-free force dual to $E$. The first issue is therefore the bilinear form coercivity and the function trace lifting properties; that is, given appropriate combinations of $E$ and its normal derivatives, is it possible to find a divergence-free field $E$ in $H^2$ whose trace on the boundary corresponds to these values?

These questions are positively answered in this paper. As the first step, a study of the extension of boundary tangent and normal vectors will also be achieved. Such extensions and their differentiability properties can be found in Theorem 2.2. We emphasize that our solution method is not very standard, since it is coordinate-free and based on the extension of an orthonormal basis on the boundary, which is thus viewed as locally Euclidean, though with a triad of local, nonconstant, basis vectors. Our core result, Theorem 3.10, states that one can find $F \in H^2_0(\Omega)$ with prescribed divergence in $\Omega$. Its main application for our purposes is about trace boundary lifting for solenoidal fields in $H^2(\Omega)$. The exact statement can be found in Theorem 3.12. Then, in section 4, combining this latter result with the bilinear form coercivity in $H^2_0(\Omega)$ (namely, Theorem 3.9), existence and uniqueness of the nonhomogeneous boundary value problem $\operatorname{inc}(\mathbb{M}\operatorname{inc} E) = \mathbb{G}$ follow in a standard way. Finally, with a view to performing topological sensitivity analysis in future work in the spirit of [2, 3], transmission conditions are identified by means of an appropriate Gauss–Green formula. They are given in Theorem 4.4.

To the best of our knowledge, our most substantial auxiliary results, namely Theorems 3.10 and 3.12, are not found elsewhere in the literature and seem to be of the utmost importance for a broad range of applications, far exceeding our study of the incompatibility operator. Let us emphasize that the incompatibility operator and its physical interpretation in dislocation modeling must be considered here as one possible application, which was the motivation for addressing this problem, but Theorems 3.10 and 3.12 show a level of generality which we believe renders their study useful to a large community of mathematicians working in applied sciences. Let us also stress the importance of boundary lifting in numerical analysis, in particular, in the finite elements methods [5, 11].

To conclude, the proper boundary value problem is discussed in section 4, while its application in dislocation models is proposed in section 5.

**Physical meaning of the model field *E* in elasticity.** The first physical interpretation of the variable $E$ is the field $F$ in Beltrami decomposition (1.2). Let the stress $\sigma$ be given, and define the elastic strain as $\epsilon := \mathbb{A}^{-1}\sigma$, where $\mathbb{A}$ is the assumed constant elasticity tensor, i.e., $\mathbb{A} = 2\mu\mathbb{I}_4 + \lambda\mathbb{I}_2\otimes\mathbb{I}_2$, with $\lambda$ and $\mu$ the Lamé coefficients. Equilibrium reads as

$$\begin{cases} -\operatorname{div}\ \sigma = -\operatorname{div}\mathbb{A}\epsilon = 0 & \text{in} \quad \Omega, \\ \sigma N = g & \text{on} \quad \partial\Omega, \end{cases} \tag{1.5}$$

and from (1.2) is rewritten as

$$\begin{cases} -\operatorname{div}(\mathbb{A}\nabla^S u) = \mathcal{F}_{\Lambda_\mathcal{L}} := \lambda\nabla\operatorname{tr}(\operatorname{inc} F) & \text{in} \quad \Omega, \\ (\mathbb{A}\nabla^S u)N = g - \lambda\operatorname{tr}(\operatorname{inc} F)N = g & \text{on} \quad \partial\Omega, \end{cases} \tag{1.6}$$

recalling the solenoidal property of inc $F$. Therefore, $u$ is called the generalized displacement field. By virtue of (1.3) with a suitable choice of boundary conditions, one also has

$$\begin{cases} \text{inc inc } F = \text{inc } \epsilon = \text{Curl } \kappa \quad \text{in} \quad \Omega, \\ F = 0 \quad \text{on} \quad \partial\Omega, \\ (\partial_N F \times N)^t \times N = 0 \quad \text{on} \quad \partial\Omega. \end{cases} \tag{1.7}$$

Systems (1.6) and (1.7) are also discussed in [21].

Moreover, recall (1.2), and take $E = \epsilon^0 :=$ inc $F$. By (1.3), the symmetric and solenoidal field $\epsilon^0$ is the part of the elastic strain which plays a role in dislocations. Thus it may be called the dislocation-induced elastic strain. Let us now address the question of a thermodynamical setting in our elastic body with dislocations in which the free energy would depend on internal defect variables such as $\epsilon^0$ and $\kappa$. Considering a high-order model in the spirit of [4], which involves the derivatives of $\kappa$ in the form of its curl, by (1.3) the free energy is then a function of $\epsilon^0$ and inc $\epsilon^0$. Now, if a quadratic model is proposed (we refer the reader again to [4]), one would naturally consider terms such as $\frac{1}{2}\mathbb{M}\,\text{inc }\epsilon^0 \cdot \text{inc }\epsilon^0$ in the free energy, with $\mathbb{M}$ a positive-definite fourth-rank tensor whose components are related to material properties of the crystal and of the dislocations. Therefore, one is led to study variational problems of the form

$$\inf_{\epsilon=\nabla^S u+\epsilon^0} \int_\Omega \left(\frac{1}{2}\mathbb{A}\nabla^S u \cdot \nabla^S u + \frac{1}{2}\mathbb{M}\,\text{inc }\epsilon^0 \cdot \text{inc }\epsilon^0\right) dx - \int_\Omega \mathbb{G} \cdot \epsilon dx,$$

where $\mathbb{G}$ is a body force that works against the total strain $\epsilon$. Of course, surface forces could also be incorporated, as discussed in section 5. If $\mathbb{G}$ is now decomposed as $\mathbb{G} = \nabla^S w + \mathbb{G}^0$, one has

$$\int_\Omega \mathbb{G} \cdot \epsilon dx = \int_\Omega \left(-\text{div } \nabla^S w \cdot u + \mathbb{G}^0 \cdot \epsilon^0\right) dx,$$

where all boundary terms have again been dropped for simplicity. Thus, $f := -\text{div } \nabla^S w$ is recognized as a body force that works against the displacement, while $\mathbb{G}^0$ works against the solenoidal part of the strain. Moreover, the minimizations with respect to $u$ and $\epsilon^0$ become uncoupled. The former provides the standard linear elasticity equations, and the latter formally yields

$$\text{inc } (\mathbb{M}\,\text{inc }\epsilon^0) = \mathbb{G}^0.$$

With these two examples of physical fields $E$, whose incompatibility plays a central role, let us now begin the mathematical analysis.

**Notation and conventions.** Let $\Omega$ be a bounded domain of $\mathbb{R}^3$ with smooth boundary $\partial\Omega$. By smooth we mean $C^\infty$, but this assumption could be considerably weakened. Let $\mathbb{M}^3$ denote the space of square $3 \times 3$ matrices, and $\mathbb{S}^3$ the space of symmetric $3 \times 3$ matrices. Divergence, curl, incompatibility, and cross product with second-order tensors are defined componentwise as follows with the summation convention on repeated indices. Here, $E$ and $T$ are second-rank tensors, $N$ is a vector, and $\epsilon$ is the Levi-Civita third-rank tensor. One has

$$\begin{aligned}
(\text{div } E)_i &:= \partial_j E_{ij}, \\
(\text{Curl } T)_{ij} &:= (\nabla \times T)_{ij} = \epsilon_{jkm} \partial_k T_{im}, \\
(\text{inc } E)_{ij} &:= (\text{Curl Curl}^{\text{t}} E)_{ij} = \epsilon_{ikm} \epsilon_{jln} \partial_k \partial_l E_{mn}, \\
(N \times T)_{ij} &:= -(T \times N)_{ij} = \epsilon_{jkm} N_k T_{im}, \\
(E \times T)_{ijk} &:= \epsilon_{jmn} E_{im} T_{kn}, \\
\text{tr } (E \times T)_j &:= \epsilon_{jmn} E_{pm} T_{pn}.
\end{aligned}$$

## 2. Extension and differentiation of the normal and tangent vectors to a surface.

**2.1. Signed distance function and extended unit normal.** We denote by $N_{\partial\Omega}$ the outward unit normal to $\partial\Omega$, and by $b$ the signed distance to $\partial\Omega$, i.e.,

$$b(x) = \begin{cases} \text{dist}(x, \partial\Omega) & \text{if } x \notin \Omega, \\ -\text{dist}(x, \partial\Omega) & \text{if } x \in \Omega. \end{cases}$$

We recall the following results (see [8, Chap. 5, Thms. 3.1 and 4.3]).

THEOREM 2.1. *There exists an open neighborhood $W$ of $\partial\Omega$ such that*

1. *$b$ is smooth in $W$;*
2. *every $x \in W$ admits a unique projection $p_{\partial\Omega}(x)$ onto $\partial\Omega$;*
3. *this projection satisfies*

$$p_{\partial\Omega}(x) = x - \frac{1}{2} \nabla b^2(x), \qquad x \in W; \tag{2.1}$$

4. *it holds that*

$$\nabla b(x) = N_{\partial\Omega}(p_{\partial\Omega}(x)), \qquad x \in W.$$

In particular, this latter property shows that $\nabla b(x) = N_{\partial\Omega}(x)$ for all $x \in \partial\Omega$ and $|\nabla b(x)| = 1$ for all $x \in W$. Therefore, we define the extended unit normal by

$$N(x) := \nabla b(x) = N_{\partial\Omega}(p_{\partial\Omega}(x)), \qquad x \in W. \tag{2.2}$$

**2.2. Tangent vectors on $\partial\Omega$.** For all $x \in \partial\Omega$, we denote by $T_{\partial\Omega}(x)$ the tangent plane to $\partial\Omega$ at $x$, that is, the orthogonal complement of $N_{\partial\Omega}(x)$. As $\partial\Omega$ is smooth, there exists a covering of $\partial\Omega$ by open balls $B_1, \ldots, B_M$ of $\mathbb{R}^3$ such that, for each index $k$, two smooth vector fields $\tau^A_{\partial\Omega}, \tau^B_{\partial\Omega}$ can be constructed on $\partial\Omega \cap B_k$, where, for all $x \in \partial\Omega \cap B_k$, $(\tau^A_{\partial\Omega}(x), \tau^B_{\partial\Omega}(x))$ is an orthonormal basis of $T_{\partial\Omega}(x)$. In what follows, the index $k$ will be implicitly considered as fixed, and the restriction to $B_k$ will be omitted. In fact, for our needs, global properties and constructions will be easily obtained from local ones through a partition of unity subordinate to the covering.

Using that the Jacobian matrix $DN(x) = D^2 b(x)$ of $N(x)$ is symmetric, differentiating the equality $|N(x)|^2 = 1$ entails

$$\partial_N N(x) = DN(x) N(x) = 0, \qquad x \in W. \tag{2.3}$$

In other words, $N(x)$ is an eigenvector of $DN(x)$ for the eigenvalue 0. For all $x \in \partial\Omega$, the system $(\tau^A_{\partial\Omega}(x), \tau^B_{\partial\Omega}(x), N_{\partial\Omega}(x))$ is an orthonormal basis of $\mathbb{R}^3$. In this basis, $DN(x)$ takes the form

$$DN(x) = \begin{pmatrix} \kappa^A_{\partial\Omega}(x) & \xi_{\partial\Omega}(x) & 0 \\ \xi_{\partial\Omega}(x) & \kappa^B_{\partial\Omega}(x) & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad x \in \partial\Omega, \tag{2.4}$$

where $\kappa^A_{\partial\Omega}$, $\kappa^B_{\partial\Omega}$, and $\xi$ are smooth scalar fields defined on $\partial\Omega$.

If $R \in \{A, B\}$, we denote by $R^*$ the complementary index of $R$; that is, $R^* = B$ if $R = A$, and $R^* = A$ if $R = B$.

**2.3. Extended tangent vectors.** Let $d$ be defined in $W$ by

$$d = \left(1 + b\ \kappa^A_{\partial\Omega} \circ p_{\partial\Omega}\right)\left(1 + b\ \kappa^B_{\partial\Omega} \circ p_{\partial\Omega}\right) - \left(b\ \xi_{\partial\Omega} \circ p_{\partial\Omega}\right)^2.$$

Possibly adjusting $W$ so that $d(x) > 0$ for all $x \in W$, we define the following in $W$:

$$\begin{aligned} \tau^R &= \tau^R_{\partial\Omega} \circ p_{\partial\Omega}, \qquad R = A, B, \\ \kappa^R &= d^{-1}\left((1 + b\ \kappa^{R^*}_{\partial\Omega} \circ p_{\partial\Omega})(\kappa^R_{\partial\Omega} \circ p_{\partial\Omega}) - b\ (\xi_{\partial\Omega} \circ p_{\partial\Omega})^2\right), \qquad R = A, B, \\ \xi &= d^{-1}\xi_{\partial\Omega} \circ p_{\partial\Omega}, \\ \kappa &= \kappa^A + \kappa^B, \\ \gamma^R &= \operatorname{div}\ \tau^R, \qquad R = A, B. \end{aligned} \tag{2.5}$$

Obviously, for each $x \in W$, the triple $(\tau^A(x), \tau^B(x), N(x))$ forms an orthonormal basis of $\mathbb{R}^3$. Next, we compute the normal and tangential derivatives of these vectors. We denote the tangential derivative $\partial_{\tau^R}$ by $\partial_R$ for simplicity. Specifically, $\partial_R g(x)$ is the differential of the vector $g$ at $x$ in the direction $\tau^R$, viz.,

$$\partial_R g(x) := Dg(x)\tau^R(x).$$

THEOREM 2.2. *It holds in $W$ that*

$$\partial_N \tau^R = 0, \tag{2.6}$$

$$\partial_R N = \kappa^R \tau^R + \xi \tau^{R^*}, \tag{2.7}$$

$$\partial_R \tau^R = -\kappa^R N - \gamma^{R^*} \tau^{R^*}, \tag{2.8}$$

$$\partial_{R^*} \tau^R = \gamma^R \tau^{R^*} - \xi N, \tag{2.9}$$

$$\operatorname{div}\ N = \operatorname{tr}\ DN = \Delta b = \kappa. \tag{2.10}$$

*Proof.* Differentiating (2.1) in the direction $h \in \mathbb{R}^3$ and using (2.2) yields

$$Dp_{\partial\Omega}(x)h = h - (N(x) \cdot h)N(x) - b(x)DN(x)h.$$

Choosing $h = N(x)$ and using (2.3) entails $Dp_{\partial\Omega}(x)N(x) = 0$; then (2.6) follows from (2.5). Choosing now $h = \tau^R(x)$ gives

$$Dp_{\partial\Omega}(x)\tau^R(x) = \tau^R(x) - b(x)DN(x)\tau^R(x). \tag{2.11}$$

Differentiating $N(x) = N(p_{\partial\Omega}(x))$ in the direction $\tau^R(x)$, one obtains using (2.11)

$$\left(I + b(x)DN(p_{\partial\Omega}(x))\right)\partial_R N(x) = DN(p_{\partial\Omega}(x))\tau^R(x). \tag{2.12}$$

Plugging (2.4) into (2.12) yields (2.7). Differentiating the relations $\tau^R(x) \cdot N(x) = 0$, $|\tau^R(x)|^2 = 1$, and $\tau^R(x) \cdot \tau^{R^*}(x) = 0$ in the direction $\tau^R(x)$ yields $\partial_R \tau^R(x) \cdot N(x) = -\kappa^R(x)$, $\partial_R \tau^R(x) \cdot \tau^R(x) = 0$, and $\partial_R \tau^R(x) \cdot \tau^{R^*}(x) + \partial_R \tau^{R^*}(x) \cdot \tau^R(x) = 0$, respectively. From

$$\operatorname{div}\ \tau^R(x) = \partial_R \tau^R(x) \cdot \tau^R(x) + \partial_{R^*} \tau^R(x) \cdot \tau^{R^*}(x) + \partial_N \tau^R(x) \cdot N(x)$$

and the preceding relations, one infers

$$\gamma^R(x) = \partial_{R^*}\tau^R(x)\cdot\tau^{R^*}(x) = -\partial_{R^*}\tau^{R^*}(x)\cdot\tau^R(x). \tag{2.13}$$

This leads to (2.8). Differentiating $\tau^R(x)\cdot N(x) = 0$ and $|\tau^R(x)|^2 = 1$ in the direction $\tau^{R^*}(x)$ yields $\partial_{R^*}\tau^R(x)\cdot N(x) = -\xi(x)$ and $\partial_{R^*}\tau^R(x)\cdot\tau^R(x) = 0$. With the help of (2.13) one arrives at (2.9). Finally, (2.10) is a straightforward consequence of (2.7) and the definitions. ∎

Corollary 2.3. *If $f$ is twice differentiable in $\Omega$, it holds that*

$$\partial_R\partial_N f = \partial_N\partial_R f + \kappa^R\partial_R f + \xi\partial_{R^*}f. \tag{2.14}$$

*Proof.* We have on the one hand

$$\begin{aligned}\partial_R\partial_N f &= \partial_R(\nabla f\cdot N)\\ &= \partial_R\nabla f\cdot N + \nabla f\cdot(\kappa^R\tau^R + \xi\tau^{R^*})\\ &= D^2 f\tau^R\cdot N + \kappa^R\partial_R f + \xi\partial_{R^*}f,\end{aligned}$$

and on the other hand

$$\partial_N\partial_R f = \partial_N(\nabla f\cdot\tau^R) = \partial_N\nabla f\cdot\tau^R = D^2 fN\cdot\tau^R.$$

One concludes the proof using the standard Schwarz lemma. ∎

**2.4. Divergence expression in the local basis.** Let us decompose a $3\times 3$ symmetric matrix $E$ in the local basis $(\tau^A, \tau^B, N)$:

$$E_{ij} = E_{NN}N_iN_j + \sum_{R=A,B} E_{NR}(N_i\tau^R_j + N_j\tau^R_i) + \sum_{R=A,B} E_{RR}\tau^R_i\tau^R_j + E_{AB}(\tau^A_i\tau^B_j + \tau^A_j\tau^B_i). \tag{2.15}$$

Using Theorem 2.2 we obtain

$$\begin{aligned}\partial_j E_{ij} &= \partial_j E_{NN}N_iN_j + E_{NN}(\partial_N N_i + \kappa N_i)\\ &\quad + \sum_R \left[\partial_j E_{NR}(N_i\tau^R_j + N_j\tau^R_i) + E_{NR}(\partial_R N_i + N_i\partial_j\tau^R_j + \partial_j N_j\tau^R_i + \partial_N\tau^R_i)\right]\\ &\quad + \sum_R \left[\partial_j E_{RR}\tau^R_i\tau^R_j + E_{RR}(\partial_R\tau^R_i + \tau^R_i\partial_j\tau^R_j)\right]\\ &\quad + \partial_j E_{AB}(\tau^A_i\tau^B_j + \tau^A_j\tau^B_i) + E_{AB}(\partial_B\tau^A_i + \tau^A_i\partial_j\tau^B_j + \tau^B_i\partial_j\tau^A_j + \partial_A\tau^B_i)\\ &= \partial_N E_{NN}N_i + E_{NN}\kappa N_i\\ &\quad + \sum_R \Big[\partial_R E_{NR}N_i + \partial_N E_{NR}\tau^R_i + E_{NR}(\kappa^R\tau^R_i + \xi\tau^{R^*}_i + \gamma^R N_i + \kappa\tau^R_i)\Big]\\ &\quad + \sum_R \Big[\partial_R E_{RR}\tau^R_i + E_{RR}(-\kappa^R N_i - \gamma^{R^*}\tau^{R^*}_i + \gamma^R\tau^R_i)\Big]\\ &\quad + \partial_B E_{AB}\tau^A_i + \partial_A E_{AB}\tau^B_i + E_{AB}(\gamma^A\tau^B_i - \xi N_i + \gamma^B\tau^A_i + \gamma^A\tau^B_i + \gamma^B\tau^A_i - \xi N_i)\\ &= N_i\bigg(\partial_N E_{NN} + \kappa E_{NN} + \sum_R\big(\partial_R E_{NR} + \gamma^R E_{NR} - \kappa^R E_{RR}\big) - 2\xi E_{AB}\bigg)\\ &\quad + \sum_R \bigg[\tau^R_i\big(\partial_N E_{NR} + (\kappa + \kappa^R)E_{NR} + \xi E_{NR^*} + \partial_R E_{RR}\\ &\qquad + \gamma^R E_{RR} - \gamma^R E_{R^*R^*} + \partial_{R^*}E_{AB} + 2\gamma^{R^*}E_{AB}\big)\bigg].\end{aligned}$$

Hence

(2.16)
$$\operatorname{div} E = \Big(\partial_N E_{NN} + \kappa E_{NN} + \sum_R (\partial_R E_{NR} + \gamma^R E_{NR} - \kappa^R E_{RR}) - 2\xi E_{AB}\Big) N + \sum_R \Big(\partial_N E_{NR} + (\kappa + \kappa^R) E_{NR} + \xi E_{NR^*} + \partial_R E_{RR} + \gamma^R E_{RR} - \gamma^R E_{R^*R^*} + \partial_{R^*} E_{AB} + 2\gamma^{R^*} E_{AB}\Big)\tau^R.$$

## 3. Function spaces.

**3.1. Definitions and basic properties.** Let $\Gamma_0$ be a subset of $\partial\Omega$ which is not everywhere flat and has nonzero two-dimensional Hausdorff measure. Define

$$\begin{aligned}
H_{\mathrm{curl}}(\Omega, \mathbb{M}^3) &:= \{E \in L^2(\Omega, \mathbb{M}^3) : \text{Curl } E \in L^2(\Omega, \mathbb{M}^3)\},\\
H_{\mathrm{inc}}(\Omega, \mathbb{S}^3) &:= \{E \in L^2(\Omega, \mathbb{S}^3) : \text{inc } E \in L^2(\Omega, \mathbb{S}^3)\},\\
\mathcal{H}(\Omega) &:= \{E \in H^2(\Omega, \mathbb{S}^3) : \text{div } E = 0\},\\
\mathcal{H}_0(\Omega) &:= \{E \in \mathcal{H}(\Omega) :\ E = (\partial_N E \times N)^t \times N = 0 \text{ on } \partial\Omega\},\\
\mathcal{H}_{\Gamma_0}(\Omega) &:= \{E \in \mathcal{H}(\Omega) :\ E = (\partial_N E \times N)^t \times N = 0 \text{ on } \Gamma_0\},\\
\tilde{H}^1_0(\Omega, \mathbb{R}^3) &:= \left\{u \in H^1_0(\Omega, \mathbb{R}^3) : \int_\Omega u dx = 0\right\},\\
\tilde{H}^{3/2}(\partial\Omega, \mathbb{S}^3) &:= \left\{E \in H^{3/2}(\partial\Omega, \mathbb{S}^3) : \int_{\partial\Omega} E N dS(x) = 0\right\}.
\end{aligned}$$

Given $\mathbb{E} \in \tilde{H}^{3/2}(\partial\Omega; \mathbb{S}^3)$ and $\mathbb{F} \in H^{1/2}(\partial\Omega; \mathbb{S}^3)$ such that $\int_{\partial\Omega} \mathbb{E} N dS(x) = 0$ and $\mathbb{F}N = 0$, we define the affine spaces

(3.1) $$\mathcal{H}_{\mathbb{E},\mathbb{F}}(\Omega) := \{E \in \mathcal{H}(\Omega) :\ E = \mathbb{E},\ (\partial_N E \times N)^t \times N = \mathbb{F} \text{ on } \partial\Omega\}$$

and

(3.2) $$\mathcal{H}_{\mathbb{E},\mathbb{F};\Gamma_0}(\Omega) := \{E \in \mathcal{H}(\Omega) :\ E = \mathbb{E},\ (\partial_N E \times N)^t \times N = \mathbb{F} \text{ on } \Gamma_0\}.$$

Obviously, in the latter case, it suffices that $\mathbb{E}$ and $\mathbb{F}$ be defined on $\Gamma_0$, and the condition $\int_{\partial\Omega} \mathbb{E} N dS(x) = 0$ is not restrictive whenever $\Gamma_0 \subset\subset \partial\Omega$. The spaces $\mathcal{H}(\Omega)$, $\mathcal{H}_0(\Omega)$ and the above affine spaces are naturally endowed with the Hilbertian structure of $H^2(\Omega, \mathbb{S}^3)$.

LEMMA 3.1. *For all $E \in H^2(\Omega, \mathbb{S}^3)$ it holds in $W$ that*

$$\text{Curl}^{\mathrm{t}}\ E \times N = -\left(\partial_N E \times N\right)^t \times N + \left(\sum_R \tau^R \times \partial_R E\right)^t \times N \qquad \textit{on } \partial\Omega.$$

*Proof.* We compute componentwise

$$\begin{aligned}
-[\text{Curl}^{\mathrm{t}}\ E \times N]_{mq} &= \epsilon_{jqv} N_v \epsilon_{mln} \partial_l E_{jn}\\
&= \epsilon_{jqv} N_v \epsilon_{mln} N_l \partial_N E_{jn} + \epsilon_{jqv} N_v \epsilon_{mln} \sum_R \tau^R_l \partial_R E_{jn}\\
&= \left((\partial_N E \times N)^t \times N\right)_{mq} - \left(\left(\sum_R \tau^R \times \partial_R E\right)^t \times N\right)_{mq},
\end{aligned}$$

proving the result. ∎

LEMMA 3.2. *For all $V \in H^1(\Omega, \mathbb{R}^3)$ it holds in $W$ that*

$$\text{Curl } V \cdot N = \partial_A V_B - \partial_B V_A - \gamma^B V_A + \gamma^A V_B.$$

*Proof.* We have

$$\begin{aligned}
\text{Curl } V \cdot N &= \epsilon_{ijk} N_i \partial_j V_k \\
&= \epsilon_{ijk} N_i \left(\partial_N V_k N_j + \partial_A V_k \tau_j^A + \partial_B V_k \tau_j^B\right) \\
&= \partial_A V \cdot \tau^B - \partial_B V \cdot \tau^A \\
&= \partial_A (V_A \tau^A + V_B \tau^B + V_N N) \cdot \tau^B - \partial_B (V_A \tau^A + V_B \tau^B + V_N N) \cdot \tau^A \\
&= \partial_A V_B + (V_A \partial_A \tau^A + V_B \partial_A \tau^B + V_N \partial_A N) \cdot \tau^B \\
&\quad - \partial_B V_A - (V_A \partial_B \tau^A + V_B \partial_B \tau^B + V_N \partial_B N) \cdot \tau^A.
\end{aligned}$$

Then one concludes the proof using Theorem 2.2. ∎

LEMMA 3.3. *Every $E \in \mathcal{H}_0(\Omega)$ satisfies*

$$\text{div Curl}^{\text{t}} E = 0 \quad \text{in } \Omega, \tag{3.3}$$

$$\text{Curl}^{\text{t}} E \times N = 0 \quad \text{on } \partial\Omega, \tag{3.4}$$

$$\partial_N E = 0 \quad \text{on } \partial\Omega. \tag{3.5}$$

*Proof.* One has

$$[\text{div Curl}^{\text{t}} E]_i = \epsilon_{ikm} \partial_j \partial_k E_{jm} = \epsilon_{ikm} \partial_k \partial_j E_{mj} = 0$$

and $\text{Curl}^{\text{t}} E \times N = 0$ by Lemma 3.1.

From (2.16) one infers that on $\partial\Omega$

$$\partial_N E_{NN} = 0, \qquad \partial_N E_{NR} = 0.$$

Therefore (2.15) entails

$$\partial_N E_{ij} = \sum_R \partial_N E_{RR} \tau_i^R \tau_j^R + \partial_N E_{AB} (\tau_i^A \tau_j^B + \tau_j^A \tau_i^B).$$

In the basis $(\tau^A, \tau^B, N)$ one has

$$\partial_N E = \begin{pmatrix} \partial_N E_{AA} & \partial_N E_{AB} & 0 \\ \partial_N E_{AB} & \partial_N E_{BB} & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \partial_N E \times N = \begin{pmatrix} \partial_N E_{AB} & -\partial_N E_{AA} & 0 \\ \partial_N E_{BB} & -\partial_N E_{AB} & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{3.6}$$

$$0 = (\partial_N E \times N)^t \times N = \begin{pmatrix} \partial_N E_{BB} & -\partial_N E_{AB} & 0 \\ -\partial_N E_{AB} & \partial_N E_{AA} & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{3.7}$$

whereby (3.5) follows. ∎

*Remark* 3.4. By the same token, for a general symmetric tensor $T$, one has in the basis $(\tau^A, \tau^B, N)$

$$T = \begin{pmatrix} T_{AA} & T_{AB} & T_{AN} \\ T_{BA} & T_{BB} & T_{BN} \\ T_{NA} & T_{NB} & T_{NN} \end{pmatrix}, \qquad T \times N = \begin{pmatrix} T_{AB} & -T_{AA} & 0 \\ T_{BB} & -T_{BA} & 0 \\ T_{NB} & -T_{NA} & 0 \end{pmatrix},$$

$$(T \times N)^t \times N = \begin{pmatrix} T_{BB} & -T_{AB} & 0 \\ -T_{AB} & T_{AA} & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{3.8}$$

*Remark* 3.5. Let $T = \text{Curl}^{\text{t}}\ E$ be such that $T \times N = 0$ on $\partial\Omega$. Then, by Remark 3.4, one has $T_{QR} = 0$ for $R = A, B$ and $Q = A, B, N$. For fixed $i$, let $V_j = T_{ij}$ in Lemma 3.2. We infer $(\text{ Curl } T)N = (\text{ inc } E)N = 0$ on $\partial\Omega$.

LEMMA 3.6 (Kozono and Yanagisawa [13] and von Wahl [24]). *Let $F \in H_{\text{curl}}(\Omega; \mathbb{M}^3)$ such that* div $F = 0$ *in* $\Omega$ *and* $F \times N = 0$ *on* $\partial\Omega$. *Then* $F \in H^1(\Omega, \mathbb{M}^3)$, *and it holds that*

$$\|\nabla F\|_{L^2(\Omega)} \leq C \|\text{Curl } F\|_{L^2(\Omega)} \tag{3.9}$$

*for some positive constant $C$ independent of $F$.*

LEMMA 3.7. *For all $E \in \mathcal{H}_0(\Omega)$ it holds that*

$$\|E\|_{H^2(\Omega)} \leq C \left( \|E\|_{L^2(\Omega)} + \|\text{Curl } E\|_{L^2(\Omega)} + \|\text{inc } E\|_{L^2(\Omega)} \right)$$

*for some positive constant $C$ independent of $E$.*

*Proof.* By Lemma 3.6 we have already

$$\|\nabla E\|_{L^2(\Omega)} \leq C \|\text{Curl } E\|_{L^2(\Omega)}.$$

Set $F = \text{Curl}^{\text{t}}\ E$. We have Curl $F \in L^2(\Omega)$, and, by Lemma 3.3, div $F = 0$ in $\Omega$ and $F \times N = 0$ on $\partial\Omega$. Hence Lemma 3.6 entails $\|\nabla F\|_{L^2(\Omega)} \leq C \|\text{Curl } F\|_{L^2(\Omega)}$, i.e.,

$$\|\partial_i \,\text{Curl}^{\text{t}}\ E\|_{L^2(\Omega)} \leq C \|\text{inc } E\|_{L^2(\Omega)}.$$

This implies

$$\|\text{Curl } \partial_i E\|_{L^2(\Omega)} \leq C \|\text{inc } E\|_{L^2(\Omega)}. \tag{3.10}$$

In addition, div $\partial_i E = \partial_i \,\text{div } E = 0$ in $\Omega$. By Lemma 3.3, $\partial_N E \times N = 0$ on $\partial\Omega$, whereby, since $E = 0$ on $\partial\Omega$, $\partial_i E \times N = 0$ on $\partial\Omega$. Therefore, by Lemma 3.6,

$$\|\partial_j \partial_i E\|_{L^2(\Omega)} \leq C \|\text{Curl } \partial_i E\|_{L^2(\Omega)}.$$

Using (3.10) we derive

$$\|\partial_j \partial_i E\|_{L^2(\Omega)} \leq C \|\text{inc } E\|_{L^2(\Omega)}, \tag{3.11}$$

and the proof is achieved. ◻

THEOREM 3.8 (Poincaré's inequality). *There exists a constant $C > 0$ such that, for each $u \in H^1(\Omega, \mathbb{R}^3)$,*

$$\|u\|_{L^2(\Omega)} \leq C \left( \|\nabla u\|_{L^2(\Omega)} + \int_{\Gamma_0} |u \times N| dS \right). \tag{3.12}$$

*Proof.* By contradiction, assume that for each $k \in \mathbb{N}$, there exists a $u_k \in H^1(\Omega; \mathbb{R}^3)$ such that

$$\|u_k\|_{L^2(\Omega)} > k \left( \|\nabla u_k\|_{L^2(\Omega)} + \int_{\Gamma_0} |u_k \times N| dS \right).$$

Defining $\dot{u}_k := u_k/\|u_k\|_{L^2(\Omega)}$, one has $\|\dot{u}_k\|_{L^2(\Omega)} = 1$, and hence (i) $\|\nabla \dot{u}_k\|_{L^2(\Omega)} \to 0$, (ii) $\int_{\Gamma_0} |\dot{u}_k \times N| dS \to 0$ as $k \to \infty$. By (i) and Rellich's theorem there exists $v \in H^s(\Omega, \mathbb{R}^3)$, $1/2 < s < 1$, such that a nonrelabeled subsequence $\dot{u}_k \to v$ in $H^s(\Omega, \mathbb{R}^3)$, and hence by virtue of (i) and for every $\varphi \in \mathcal{D}(\Omega, \mathbb{R}^3)$,

$$\int_\Omega v \,\mathrm{div}\, \varphi dx = \lim_{k\to\infty} \int_\Omega \dot{u}_k \,\mathrm{div}\, \varphi dx = -\lim_{k\to\infty} \int_\Omega D\dot{u}_k \varphi dx = 0,$$

whereby $\nabla v = 0$, meaning that $v$ is a constant vector. Condition (ii) now implies that $\int_{\Gamma_0} |\dot{u}_k \times N| dS \to \int_{\Gamma_0} |v \times N| dS = 0$ as $k \to \infty$; i.e., $v \times N = 0$ and thus $v$ is parallel to $N$, which is not constant and of unit length, and hence $v = 0$, a contradiction, since $\|v\|_{L^2(\Omega)} = 1$. ∎

THEOREM 3.9 (coercivity). *There exists a positive constant $C$ such that, for each $E \in \mathcal{H}_0(\Omega)$,*

$$\|E\|_{H^2(\Omega)} \le C \|\mathrm{inc}\; E\|_{L^2(\Omega)}. \tag{3.13}$$

*Proof.* By Theorem 3.8, the tensor counterpart of (3.12) reads as

$$\|F\|_{L^2(\Omega)} \le C \left( \|\nabla F\|_{L^2(\Omega)} + \int_{\Gamma_0} |F \times N| dS \right)$$

for all $F \in H^1(\Omega, \mathbb{M}^3)$. By Lemma 3.3, $\mathrm{div}\, \mathrm{Curl}^{\mathrm{t}}\; E = 0$ in $\Omega$ and $\mathrm{Curl}^{\mathrm{t}}\; E \times N = 0$ on $\partial\Omega$. Hence by Lemma 3.6 and again by Theorem 3.8, one has (with the nonrelabeled constant $C > 0$)

$$\begin{aligned} \|E\|_{L^2(\Omega)} &\le C\|\nabla E\|_{L^2(\Omega)} \le C\|\mathrm{Curl}\; E\|_{L^2(\Omega)} = C\|\mathrm{Curl}^{\mathrm{t}}\; E\|_{L^2(\Omega)} \\ &\le C\|\nabla\, \mathrm{Curl}^{\mathrm{t}}\; E\|_{L^2(\Omega)} \le C\|\mathrm{Curl}\;\; \mathrm{Curl}^{\mathrm{t}}\; E\|_{L^2(\Omega)} = C\|\mathrm{inc}\; E\|_{L^2(\Omega)}. \end{aligned}$$

The proof is completed using Lemma 3.7. ∎

**3.2. Lifting of boundary traces.**

THEOREM 3.10. *Let $g \in \tilde{H}^1_0(\Omega, \mathbb{R}^3)$. Then there exists $U \in H^2_0(\Omega, \mathbb{S}^3)$ such that $\mathrm{div}\; U = g$.*

*Proof.*

*Step* 1. Let $v \in H^1(\Omega, \mathbb{R}^3)$ be a solution (unique up to a rigid motion) of

$$\begin{cases} -\,\mathrm{div}\; \nabla^s v = g & \text{in } \Omega, \\ \nabla^s v N = 0 & \text{on } \partial\Omega, \end{cases}$$

and set $V = \nabla^s v$. By elliptic regularity, $v \in H^3(\Omega)$; thus $V \in H^2(\Omega)$. We have $\mathrm{div}\; V = g$ in $\Omega$ and $VN = 0$ on $\partial\Omega$.

*Step* 2. We aim at defining $U = V + W$, where $W = \mathrm{inc}\; \Psi$, $\Psi \in H^4(\Omega, \mathbb{S}^3)$, must satisfy

$$WN = 0 \quad \text{on } \partial\Omega, \tag{3.14}$$

$$W\tau^R = -V\tau^R \quad \text{on } \partial\Omega, \tag{3.15}$$

$$\partial_N W = -\partial_N V \quad \text{on } \partial\Omega. \tag{3.16}$$

We assume a priori that $\Psi \in H^4(\Omega, \mathbb{S}^3)$ satisfies

$$\Psi = \partial_N \Psi = 0 \quad \text{on } \partial\Omega. \tag{3.17}$$

Then (3.14) holds true by Remark 3.5 and Lemma 3.1. We are going to derive other conditions on $\Psi$ such that (3.15)–(3.16) are also satisfied.

*Step* 3. Now let us rewrite the traces of $V$ and $\partial_N V$ on $\partial\Omega$ in the local basis $(\tau^A, \tau^B, N)$ as

$$V = \begin{pmatrix} V_{AA} & V_{AB} & V_{AN} \\ V_{AB} & V_{BB} & V_{BN} \\ V_{AN} & V_{BN} & V_{NN} \end{pmatrix} = \begin{pmatrix} V_{AA} & V_{AB} & 0 \\ V_{AB} & V_{BB} & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$\partial_N V = \begin{pmatrix} \partial_N V_{AA} & \partial_N V_{AB} & \partial_N V_{AN} \\ \partial_N V_{AB} & \partial_N V_{BB} & \partial_N V_{BN} \\ \partial_N V_{AN} & \partial_N V_{BN} & \partial_N V_{NN} \end{pmatrix}.$$

Assume that

$$\partial_N^2 \Psi = \begin{pmatrix} -V_{BB} & V_{AB} & 0 \\ V_{AB} & -V_{AA} & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{3.18}$$

and

$$\partial_N^3 \Psi = \begin{pmatrix} (-\partial_N + 2\kappa^A)V_{BB} - 2\xi V_{AB} & (\partial_N - \kappa)V_{AB} + \xi(V_{AA} + V_{BB}) & 0 \\ (\partial_N - \kappa)V_{AB} + \xi(V_{AA} + V_{BB}) & (-\partial_N + 2\kappa^B)V_{AA} - 2\xi V_{AB} & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{3.19}$$

Let us compute the components of the vector $W\tau^R$.

• For $W\tau^R \cdot N$, it holds that

$$W\tau^R \cdot N : W\tau^R \cdot N = WN \cdot \tau^R = 0 = -V_{RN}. \tag{3.20}$$

• For $W\tau^R \cdot \tau^R$, we compute componentwise

$$\begin{aligned} W_{ij}\tau_j^A\tau_i^A &= \epsilon_{ikm}\tau_i^A \epsilon_{jln}\tau_j^A \partial_k \partial_l \Psi_{mn} \\ &= \epsilon_{ikm}\tau_i^A \epsilon_{jln} \partial_k(\tau_j^A \partial_l \Psi_{mn}) - \epsilon_{ikm}\tau_i^A \epsilon_{jln}(\partial_k \tau_j^A)\partial_l \Psi_{mn}. \end{aligned}$$

The last term of the right-hand side vanishes by (3.17); hence

$$\begin{aligned} W_{ij}\tau_j^A\tau_i^A &= \epsilon_{ikm}\tau_i^A \partial_k(\epsilon_{jln}\tau_j^A \partial_l \Psi_{mn}) \\ &= \epsilon_{ikm}\tau_i^A(\tau_k^B \partial_B + N_k \partial_N)\left(\epsilon_{jln}\tau_j^A(\tau_l^B \partial_B + N_l \partial_N)\Psi_{mn}\right) \\ &= (N_m \partial_B - \tau_m^B \partial_N)(N_n \partial_B - \tau_n^B \partial_N)\Psi_{mn} \\ &= N_m \partial_B(N_n \partial_B \Psi_{mn}) - N_m \partial_B(\tau_n^B \partial_N \Psi_{mn}) - \tau_m^B \partial_N(N_n \partial_B \Psi_{mn}) \\ &\quad + \tau_m^B \partial_N(\tau_n^B \partial_N \Psi_{mn}), \end{aligned}$$

which again by (3.17) yields

$$W_{ij}\tau_j^A\tau_i^A = -\tau_m^B N_n \partial_N(\partial_B \Psi_{mn}) + \tau_m^B \tau_n^B \partial_N(\partial_N \Psi_{mn});$$

that is, by (3.17) and (2.14),

$$W_{ij}\tau_j^A\tau_i^A = \partial_N \partial_N(\tau_m^B \tau_n^B \Psi_{mn}) = \partial_N^2 \Psi_{BB}. \tag{3.21}$$

Similarly, it holds that $W_{ij}\tau_j^B\tau_i^B = \partial_N^2\Psi_{AA}$.

• Now, consider $W\tau^R\cdot\tau^{R^*}$ and compute componentwise

$$\begin{aligned}W_{ij}\tau_j^A\tau_i^B &= \epsilon_{ikm}\tau_i^B\epsilon_{jln}\tau_j^A\partial_k\partial_l\Psi_{mn} = \epsilon_{ikm}\tau_i^B\partial_k(\epsilon_{jln}\tau_j^A\partial_l\Psi_{mn})\\ &= \epsilon_{ikm}\tau_i^B(\tau_k^A\partial_A + N_k\partial_N)\left(\epsilon_{jln}\tau_j^A(\tau_l^B\partial_B + N_l\partial_N)\Psi_{mn}\right)\\ &= (-N_m\partial_A + \tau_m^A\partial_N)(N_n\partial_B - \tau_n^B\partial_N)\Psi_{mn}\\ &= -N_m\partial_A(N_n\partial_B\Psi_{mn}) + N_m\partial_A(\tau_n^B\partial_N\Psi_{mn}) + \tau_m^A\partial_N(N_n\partial_B\Psi_{mn})\\ &\quad - \tau_m^A\partial_N(\tau_n^B\partial_N\Psi_{mn}).\end{aligned}$$

By (3.17) and (2.14), this reads as

$$W_{ij}\tau_j^A\tau_i^B = -\partial_N^2\Psi_{AB}. \tag{3.22}$$

Thus (3.15) is satisfied by (3.18) and (3.20)–(3.22).

*Step* 4. Let us compute $\partial_N W$.

• We first compute $\partial_N W\tau^R\cdot\tau^R$. Recall that from Corollary 2.3, one has

$$\partial_{NA} = \partial_{AN} - \kappa^A\partial_A - \xi\partial_B. \tag{3.23}$$

By mere projections, we have

$$\begin{aligned}\partial_N W_{ij}\tau_j^A\tau_i^A &= \partial_N\left[\epsilon_{ikm}\epsilon_{jln}\tau_i^A\tau_j^A\partial_k\partial_l\Psi_{mn}\right]\\ &= \partial_N\left[\epsilon_{ikm}\epsilon_{jln}\tau_i^A\tau_j^A(\tau_k^A\partial_A + \tau_k^B\partial_B + N_k\partial_N)(\tau_l^A\partial_A + \tau_l^B\partial_B + N_l\partial_N)\Psi_{mn}\right]\\ &= \partial_N\left[\epsilon_{ikm}\tau_i^A(\tau_k^A\partial_A + \tau_k^B\partial_B + N_k\partial_N)\left(\epsilon_{jln}\tau_j^A(\tau_l^A\partial_A + \tau_l^B\partial_B + N_l\partial_N)\Psi_{mn}\right)\right.\\ &\quad\left. - \epsilon_{ikm}\epsilon_{jln}\tau_i^A(\tau_k^A\partial_A + \tau_k^B\partial_B + N_k\partial_N)\tau_j^A(\tau_l^A\partial_A + \tau_l^B\partial_B + N_l\partial_N)\Psi_{mn}\right]\\ &= \partial_N\left[(N_m\partial_B - \tau_m^B\partial_N)(N_n\partial_B - \tau_n^B\partial_N)\Psi_{mn}\right.\\ &\quad\left. - \epsilon_{jln}\left((N_m\partial_B - \tau_m^B\partial_N)\tau_j^A\right)(\tau_l^A\partial_A + \tau_l^B\partial_B + N_l\partial_N)\Psi_{mn}\right]\\ &= (N_m\partial_{NB} - \tau_m^B\partial_{NN})(N_n\partial_B - \tau_n^B\partial_N)\Psi_{mn}\\ &\quad - \epsilon_{jln}N_m\partial_B\tau_j^A(\tau_l^A\partial_{NA} + \tau_l^B\partial_{NB} + N_l\partial_{NN})\Psi_{mn}\\ &\quad - \epsilon_{jln}\left((N_m\partial_{NB} - \tau_m^B\partial_{NN})\tau_j^A\right)(\tau_l^A\partial_A + \tau_l^B\partial_B + N_l\partial_N)\Psi_{mn},\end{aligned} \tag{3.24}$$

and hence from (3.17) and (3.23), the right-hand side of (3.24) is equal to

$$(N_m\partial_{NB} - \tau_m^B\partial_{NN})(N_n\partial_B - \tau_n^B\partial_N)\Psi_{mn} - \epsilon_{jln}N_m\partial_B\tau_j^A(\tau_l^A\partial_{NA} + \tau_l^B\partial_{NB} + N_l\partial_{NN})\Psi_{mn}.$$

By (3.23), it follows that

$$\begin{aligned}\partial_N W_{ij}\tau_j^A\tau_i^A &= (N_m\partial_{BN} - \kappa^B N_m\partial_B - \xi N_m\partial_A - \tau_m^B\partial_{NN})(N_n\partial_B - \tau_n^B\partial_N)\Psi_{mn}\\ &\qquad - \epsilon_{jln}N_m(\gamma^A\tau_j^B - \xi N_j)(\tau_l^A\partial_{NA} + \tau_l^B\partial_{NB} + N_l\partial_{NN})\Psi_{mn};\end{aligned}$$

thus by virtue of (3.17) and (3.23),

$$\begin{aligned}\partial_N W_{ij}\tau_j^A\tau_i^A &= (N_m\partial_{BN} - \tau_m^B\partial_{NN})(N_n\partial_B - \tau_n^B\partial_N)\Psi_{mn}\\ &\quad - \epsilon_{jln}N_m(\gamma^A\tau_j^B - \xi N_j)(N_l\partial_{NN})\Psi_{mn}\\ &= (N_m\partial_B - \tau_m^B\partial_N)(N_n\partial_{NB} - \tau_n^B\partial_{NN})\Psi_{mn} - N_m(\gamma^A\tau_n^A)\partial_{NN}\Psi_{mn},\end{aligned}$$

and again by (3.23),

$$\partial_N W_{ij}\tau_j^A\tau_i^A = (N_m\partial_B - \tau_m^B\partial_N)(N_n\partial_{BN} - \kappa^B N_n\partial_B - \xi N_n\partial_A - \tau_n^B\partial_{NN})\Psi_{mn} - \gamma^A N_m\tau_n^A\partial_{NN}\Psi_{mn}.$$

By (3.17), this entails that

$$\begin{aligned}\partial_N W_{ij}\tau_j^A\tau_i^A = {} & N_m\partial_B(-\tau_n^B\partial_{NN})\Psi_{mn} - \tau_m^B(N_n\partial_{NBN} - \kappa^B N_n\partial_{NB} - (\partial_N\kappa^B)N_n\partial_B \\ & - \xi N_n\partial_{NA} - (\partial_N\xi)N_n\partial_A - \tau_n^B\partial_{NNN})\Psi_{mn} - \gamma^A N_m\tau_n^A\partial_{NN}\Psi_{mn},\end{aligned}$$

whereby, again using (3.17) and (3.23),

$$\begin{aligned}\partial_N W_{ij}\tau_j^A\tau_i^A = & - N_m\partial_B(\tau_n^B\partial_{NN})\Psi_{mn} - \tau_m^B(N_n\partial_{NBN} - \tau_n^B\partial_{NNN})\Psi_{mn} \\ & - \gamma^A N_m\tau_n^A\partial_{NN}\Psi_{mn} \\ = & - N_m(-\kappa^B N_n\partial_{NN} - \gamma^A\tau_n^A\partial_{NN} + \tau_n^B\partial_{BNN})\Psi_{mn} \\ & - \tau_m^B(N_n\partial_{BNN} - \tau_n^B\partial_{NNN})\Psi_{mn} - \gamma^A N_m\tau_n^A\partial_{NN}\Psi_{mn}.\end{aligned}$$

Therefore,

$$\begin{aligned}\partial_N W_{ij}\tau_j^A\tau_i^A &= \kappa^B N_m N_n\partial_{NN}\Psi_{mn} - (N_m\tau_n^B + N_n\tau_m^B)\partial_{BNN}\Psi_{mn} + \tau_m^B\tau_n^B\partial_{NNN}\Psi_{mn} \\ &= \kappa^B N_m N_n\partial_{NN}\Psi_{mn} - 2N_m\tau_n^B\partial_{BNN}\Psi_{mn} + \tau_m^B\tau_n^B\partial_{NNN}\Psi_{mn}. \qquad (3.25)\end{aligned}$$

Yet

$$\begin{aligned}\partial_B(N_m\tau_n^B) &= (\kappa^B\tau_m^B + \xi\tau_m^A)\tau_n^B + N_m(-\kappa^B N_n - \gamma^A\tau_n^A) \\ &= \kappa^B\tau_m^B\tau_n^B + \xi\tau_m^A\tau_n^B - \kappa^B N_m N_n - \gamma^A N_m\tau_n^A. \qquad (3.26)\end{aligned}$$

This yields

$$\begin{aligned}\partial_N W_{ij}\tau_j^A\tau_i^A = {} & \kappa^B N_m N_n\partial_{NN}\Psi_{mn} - 2\partial_B(N_m\tau_n^B\partial_{NN}\Psi_{mn}) \\ & + 2(\kappa^B\tau_m^B\tau_n^B + \xi\tau_m^A\tau_n^B - \kappa^B N_m N_n - \gamma^A N_m\tau_n^A)\partial_{NN}\Psi_{mn} \\ & + \tau_m^B\tau_n^B\partial_{NNN}\Psi_{mn} \\ = & - \kappa^B N_m N_n\partial_{NN}\Psi_{mn} - 2\partial_B(N_m\tau_n^B\partial_{NN}\Psi_{mn}) \\ & + 2(\kappa^B\tau_m^B\tau_n^B + \xi\tau_m^A\tau_n^B - \gamma^A N_m\tau_n^A)\partial_{NN}\Psi_{mn} + \tau_m^B\tau_n^B\partial_{NNN}\Psi_{mn} \\ = & - \kappa^B\partial_{NN}\Psi_{NN} - 2\partial_B\partial_{NN}\Psi_{BN} \\ & + 2\kappa^B\partial_{NN}\Psi_{BB} + 2\xi\partial_{NN}\Psi_{AB} - 2\gamma^A\partial_{NN}\Psi_{AN} + \partial_{NNN}\Psi_{BB},\end{aligned}$$

which implies by (3.18) and (3.19) that

$$\partial_N W_{ij}\tau_j^A\tau_i^A = - 2\kappa^B V_{AA} + 2\xi V_{AB} + \partial_N^3\Psi_{BB} = -\partial_N V_{AA}. \qquad (3.27)$$

We have obtained $\partial_N W_{AA} = -\partial_N V_{AA}$. Similarly we find $\partial_N W_{BB} = -\partial_N V_{BB}$.

• Then we compute $\partial_N W\tau^R\cdot\tau^{R^*}$. We have

$$\begin{aligned}\partial_N W_{ij}\tau_j^A\tau_i^B &= \partial_N\left[\epsilon_{ikm}\epsilon_{jln}\tau_i^B\tau_j^A\partial_k\partial_l\Psi_{mn}\right] \\ &= \partial_N\left[\epsilon_{ikm}\epsilon_{jln}\tau_i^B\tau_j^A(\tau_k^A\partial_A + \tau_k^B\partial_B + N_k\partial_N)(\tau_l^A\partial_A + \tau_l^B\partial_B + N_l\partial_N)\Psi_{mn}\right] \\ &= \partial_N\big[(-N_m\partial_A + \tau_m^A\partial_N)(N_n\partial_B - \tau_n^B\partial_N)\Psi_{mn} \\ &\quad - \epsilon_{jln}(-N_m\partial_A + \tau_m^A\partial_N)\tau_j^A(\tau_l^A\partial_A + \tau_l^B\partial_B + N_l\partial_N)\Psi_{mn}\big],\end{aligned}$$

which by virtue (3.17) is rewritten as

$$\begin{aligned}\partial_N W_{ij}\tau_j^A\tau_i^B &= (-N_m\partial_{NA} + \tau_m^A\partial_{NN})(N_n\partial_B - \tau_n^B\partial_N)\Psi_{mn}\\ &\quad + \epsilon_{jln}N_m(-\kappa^A N_j - \gamma^B\tau_j^B)(\tau_l^A\partial_{NA} + \tau_l^B\partial_{NB} + N_l\partial_{NN})\Psi_{mn}.\end{aligned}$$

Hence, by (3.17) and (3.23),

$$\begin{aligned}\partial_N W_{ij}\tau_j^A\tau_i^B &= (-N_m\partial_{AN} + \kappa^A N_m\partial_A + \xi N_m\partial_B + \tau_m^A\partial_{NN})(N_n\partial_B - \tau_n^B\partial_N)\Psi_{mn}\\ &\quad + \epsilon_{jln}N_m(-\kappa^A N_j - \gamma^B\tau_j^B)(\tau_l^A\partial_{NA} + \tau_l^B\partial_{NB} + N_l\partial_{NN})\Psi_{mn},\end{aligned}$$

and by (3.17) again, one has

$$\begin{aligned}\partial_N W_{ij}\tau_j^A\tau_i^B &= (-N_m\partial_{AN} + \tau_m^A\partial_{NN})(N_n\partial_B - \tau_n^B\partial_N)\Psi_{mn}\\ &\quad - \epsilon_{jln}N_m(\kappa^A N_j + \gamma^B\tau_j^B)N_l\partial_{NN}\Psi_{mn}\\ &= (-N_m\partial_A + \tau_m^A\partial_N)(N_n\partial_{NB} - \tau_n^B\partial_{NN})\Psi_{mn} - \gamma^B N_m\tau_n^A\partial_{NN}\Psi_{mn}. \quad (3.28)\end{aligned}$$

Again by (3.23), the right-hand side of (3.28) is equal to

$$(-N_m\partial_A + \tau_m^A\partial_N)(N_n\partial_{BN} - \kappa^B N_n\partial_B - \xi N_n\partial_A - \tau_n^B\partial_{NN})\Psi_{mn} - \gamma^B N_m\tau_n^A\partial_{NN}\Psi_{mn},$$

which by (3.17) is rewritten as

$$\begin{aligned}&N_m\partial_A\tau_n^B\partial_{NN}\psi_{mn} + \tau_m^A(N_n\partial_{NBN} - \kappa^B N_n\partial_{NB} - \xi N_n\partial_{NA} - \tau_n^B\partial_{NNN})\Psi_{mn}\\ &\quad - \gamma^B N_m\tau_n^A\partial_{NN}\Psi_{mn}.\end{aligned}$$

Therefore, (3.17) and (3.23) imply that

$$\begin{aligned}\partial_N W_{ij}\tau_j^A\tau_i^B &= N_m\partial_A(\tau_n^B\partial_{NN})\psi_{mn} + \tau_m^A(N_n\partial_{NBN} - \tau_n^B\partial_{NNN})\Psi_{mn} - \gamma^B N_m\tau_n^A\partial_{NN}\Psi_{mn}\\ &= N_m(\gamma^B\tau_n^A\partial_{NN} - \xi N_n\partial_{NN} + \tau_n^B\partial_{ANN})\Psi_{mn}\\ &\quad + \tau_m^A(N_n\partial_{BNN} - \tau_n^B\partial_{NNN})\Psi_{mn} - \gamma^B N_m\tau_n^A\partial_{NN}\Psi_{mn}\\ &= -\xi N_m N_n\partial_{NN}\Psi_{mn} + N_m\tau_n^B\partial_{ANN}\Psi_{mn} + N_n\tau_m^A\partial_{BNN}\Psi_{mn}\\ &\quad - \tau_m^A\tau_n^B\partial_{NNN}\Psi_{mn}.\end{aligned}$$

Yet,

$$\partial_A(N_m\tau_n^B) + \partial_B(N_n\tau_m^A) = \kappa\tau_m^A\tau_n^B + \xi(\tau_m^A\tau_n^A + \tau_m^B\tau_n^B) + \gamma^B N_m\tau_n^A + \gamma^A N_n\tau_m^B - 2\xi N_m N_n.$$

This yields

$$\begin{aligned}\partial_N W_{ij}\tau_j^A\tau_i^B &= -\xi N_m N_n\partial_{NN}\Psi_{mn} + \partial_A(N_m\tau_n^B\partial_{NN}\Psi_{mn}) + \partial_B(N_n\tau_m^A\partial_{NN}\Psi_{mn})\\ &\quad - \left(\kappa\tau_m^A\tau_n^B + \xi(\tau_m^A\tau_n^A + \tau_m^B\tau_n^B) + \gamma^B N_m\tau_n^A + \gamma^A N_n\tau_m^B - 2\xi N_m N_n\right)\partial_{NN}\Psi_{mn}\\ &\quad - \tau_m^A\tau_n^B\partial_{NNN}\Psi_{mn}\\ &= \xi N_m N_n\partial_{NN}\Psi_{mn} + \partial_A(N_m\tau_n^B\partial_{NN}\Psi_{mn}) + \partial_B(N_n\tau_m^A\partial_{NN}\Psi_{mn})\\ &\quad - \left(\kappa\tau_m^A\tau_n^B + \xi(\tau_m^A\tau_n^A + \tau_m^B\tau_n^B) + \gamma^B N_m\tau_n^A + \gamma^A N_n\tau_m^B\right)\partial_{NN}\Psi_{mn}\\ &\quad - \tau_m^A\tau_n^B\partial_{NNN}\Psi_{mn}\\ &= \xi\partial_{NN}\Psi_{NN} + \partial_A(\partial_{NN}\Psi_{BN}) + \partial_B(\partial_{NN}\Psi_{AN})\\ &\quad - \kappa\partial_{NN}\Psi_{AB} - \xi\partial_{NN}(\Psi_{AA} + \Psi_{BB}) - \gamma^B\partial_{NN}\Psi_{AN} - \gamma^A\partial_{NN}\Psi_{BN}\\ &\quad - \partial_{NNN}\Psi_{AB}.\end{aligned}$$

Thus, by (3.18) and (3.19), one has

$$\partial_N W_{ij}\tau_j^A\tau_i^B = -\kappa V_{AB} + \xi(V_{AA}+V_{BB}) - \partial_N^3\Psi_{AB} = -\partial_N V_{AB}.$$

• Now we address the term $\partial_N WN\cdot\tau^R$. It holds that

$$\begin{aligned}\partial_N W_{ij}N_j\tau_i^A = \partial_N\big[&(N_m\partial_B - \tau_m^B\partial_N)(\tau_n^B\partial_A - \tau_n^A\partial_B)\Psi_{mn}\\ &- \epsilon_{jln}\left((N_m\partial_B + \tau_m^B\partial_N)N_j\right)(\tau_l^A\partial_A + \tau_l^B\partial_B + N_l\partial_N)\Psi_{mn}\big],\end{aligned}$$

which by (3.17) is rewritten as

$$\begin{aligned}\partial_N W_{ij}N_j\tau_i^A = &(N_m\partial_{NB} - \tau_m^B\partial_{NN})(\tau_n^B\partial_A - \tau_n^A\partial_B)\Psi_{mn}\\ &- \epsilon_{jln}N_m\left(\partial_B N_j\right)(\tau_l^A\partial_{NA} + \tau_l^B\partial_{NB} + N_l\partial_{NN})\Psi_{mn},\end{aligned}$$

and by (3.17) and (3.23), as

$$\partial_N W_{ij}N_j\tau_i^A = (N_m\partial_{BN} - \tau_m^B\partial_{NN})(\tau_n^B\partial_A - \tau_n^A\partial_B)\Psi_{mn} - \epsilon_{jln}N_m\left(\partial_B N_j\right)N_l\partial_{NN}\Psi_{mn}.$$

The last term vanishes since $\partial_{NN}\Psi_{iN}=0$. Thus

$$\partial_N W_{ij}N_j\tau_i^A = (N_m\partial_B - \tau_m^B\partial_N)(\tau_n^B\partial_{NA} - \tau_n^A\partial_{NB})\Psi_{mn} - \gamma^A N_m\tau_n^A\partial_{NN}\Psi_{mn},$$

and by (3.23), is rewritten as

$$\begin{aligned}\partial_N W_{ij}N_j\tau_i^A = (N_m\partial_B - \tau_m^B\partial_N)(&\tau_n^B\partial_{AN} - \kappa^A\tau_n^B\partial_A - \xi\tau_n^B\partial_B - \tau_n^A\partial_{BN}\\ &+ \kappa^B\tau_n^A\partial_B + \xi\tau_n^A\partial_A)\Psi_{mn},\end{aligned}$$

which by virtue (3.17) is rewritten as

$$\begin{aligned}\partial_N W_{ij}N_j\tau_i^A &= -\tau_m^B\partial_N(\tau_n^B\partial_{AN} - \kappa^A\tau_n^B\partial_A - \xi\tau_n^B\partial_B - \tau_n^A\partial_{BN} + \kappa^B\tau_n^A\partial_B + \xi\tau_n^A\partial_A)\Psi_{mn}\\ &= -\tau_m^B(\tau_n^B\partial_{NAN} - \kappa^A\tau_n^B\partial_{NA} - \xi\tau_n^B\partial_{NB} - \tau_n^A\partial_{NBN}\\ &\qquad + \kappa^B\tau_n^A\partial_{NB} + \xi\tau_n^A\partial_{NA})\Psi_{mn}.\end{aligned}$$

Again by (3.17) and (3.23), one writes

$$\begin{aligned}\partial_N W_{ij}N_j\tau_i^A = -\tau_m^B(&\tau_n^B\partial_{ANN} - \kappa^A\tau_n^B\partial_{AN} - \xi\tau_n^B\partial_{BN} - \tau_n^A\partial_{BNN}\\ &+ \kappa^B\tau_n^A\partial_{BN} + \xi\tau_n^A\partial_{AN})\Psi_{mn},\end{aligned}$$

which, recalling (3.17), reads as

$$\begin{aligned}\partial_N W_{ij}N_j\tau_i^A &= -\tau_m^B(\tau_n^B\partial_{ANN} - \tau_n^A\partial_{BNN})\Psi_{mn}\\ &= -\tau_m^B\tau_n^B\partial_{ANN}\Psi_{mn} + \tau_m^B\tau_n^A\partial_{BNN}\Psi_{mn}.\end{aligned}$$

Yet,

$$\begin{aligned}-\partial_A(\tau_m^B\tau_n^B) + \partial_B(\tau_m^B\tau_n^A) = &-\gamma^B(\tau_m^A\tau_n^B + \tau_m^B\tau_n^A) + \xi N_m\tau_n^B - \kappa^B N_m\tau_n^A\\ &+ \gamma^A(-\tau_m^A\tau_n^A + \tau_m^B\tau_n^B),\end{aligned}$$

and hence

$$\begin{aligned}\partial_N W_{ij} N_j \tau_i^A =& - \partial_A(\tau_m^B \tau_n^B \partial_{NN}\Psi_{mn}) + \partial_B(\tau_m^B \tau_n^A \partial_{NN}\Psi_{mn}) - \gamma^A N_m \tau_n^A \partial_{NN}\Psi_{mn} \\ & - \big( - \gamma^B(\tau_m^A \tau_n^B + \tau_m^B \tau_n^A) + \xi N_m \tau_n^B - \kappa^B N_m \tau_n^A \\ & \quad + \gamma^A(-\tau_m^A \tau_n^A + \tau_m^B \tau_n^B)\big)\partial_{NN}\Psi_{mn} \\ =& - \partial_{ANN}\Psi_{BB} + \partial_{BNN}\Psi_{AB} - \gamma^A \partial_{NN}\Psi_{NA} \\ & + 2\gamma^B \partial_{NN}\Psi_{AB} - \xi\partial_{NN}\Psi_{BN} + \kappa^B \partial_{NN}\Psi_{AN} + \gamma^A \partial_{NN}(\Psi_{AA} - \Psi_{BB}).\end{aligned}$$

Therefore, (3.18) yields

$$\partial_N W_{ij} N_j \tau_i^A = \partial_A V_{AA} + \partial_B V_{AB} + 2\gamma_B V_{AB} + \gamma^A(V_{AA} - V_{BB}).$$

Since div $V = 0$ and $VN = 0$, we infer from (2.16) that

$$\partial_N V_{AN} + \partial_A V_{AA} + \gamma^A(V_{AA} - V_{BB}) + \partial_B V_{AB} + 2\gamma^B V_{AB} = 0.$$

Thus

$$\partial_N W_{AN} = -\partial_N V_{AN}.$$

Similarly, one gets $\partial_N W_{BN} = -\partial_N V_{BN}$.

- Finally, it remains to consider $\partial_N W N \cdot N$. We have

$$\begin{aligned}\partial_N W_{ij} N_j N_i = \partial_N \big[ & (\tau_m^B \partial_A - \tau_m^A \partial_B)(\tau_n^B \partial_A - \tau_n^A \partial_B)\Psi_{mn} \\ & - \epsilon_{jln}(\tau_m^B \partial_A - \tau_m^A \partial_B) N_j (\tau_l^A \partial_A + \tau_l^B \partial_B + N_l \partial_N)\Psi_{mn}\big],\end{aligned}$$

which by (3.17) is written as

$$\begin{aligned}\partial_N W_{ij} N_j N_i =& (\tau_m^B \partial_{NA} - \tau_m^A \partial_{NB})(\tau_n^B \partial_A - \tau_n^A \partial_B)\Psi_{mn} \\ & - \epsilon_{jln}(\tau_m^B \partial_A - \tau_m^A \partial_B) N_j (\tau_l^A \partial_{NA} + \tau_l^B \partial_{NB} + N_l \partial_{NN})\Psi_{mn},\end{aligned}$$

and again by (3.17), recalling (3.23), is rewritten as

$$\begin{aligned}\partial_N W_{ij} N_j N_i =& (\tau_m^B \partial_{AN} - \tau_m^A \partial_{BN})(\tau_n^B \partial_A - \tau_n^A \partial_B)\Psi_{mn} \\ & - \epsilon_{jln}(\kappa^A \tau_m^B \tau_j^A + \xi \tau_m^B \tau_j^B - \kappa^B \tau_m^A \tau_j^B - \xi \tau_m^A \tau_j^A) N_l \partial_{NN}\Psi_{mn} \\ =& (\tau_m^B \partial_A - \tau_m^A \partial_B)(\tau_n^B \partial_{NA} - \tau_n^A \partial_{NB})\Psi_{mn} \\ & - (-\kappa^A \tau_m^B \tau_n^B + \xi \tau_m^B \tau_n^A - \kappa^B \tau_m^A \tau_n^A + \xi \tau_m^A \tau_n^B)\partial_{NN}\Psi_{mn}.\end{aligned}$$

Hence, (3.17) and (3.23) imply that

$$\begin{aligned}\partial_N W_{ij} N_j N_i =& (\tau_m^B \partial_A - \tau_m^A \partial_B)(\tau_n^B \partial_{AN} - \tau_n^A \partial_{AN})\Psi_{mn} \\ & + (\kappa^A \tau_m^B \tau_n^B + \kappa^B \tau_m^A \tau_n^A - \xi \tau_m^B \tau_n^A - \xi \tau_m^A \tau_n^B)\partial_{NN}\Psi_{mn}.\end{aligned}$$

Yet, (3.17) yields

$$\partial_N W_{ij} N_j N_i = \kappa^A \partial_{NN}\Psi_{BB} + \kappa^B \partial_{NN}\Psi_{AA} - 2\xi\partial_{NN}\Psi_{AB},$$

which by (3.18) achieves the calculation, since

$$\partial_N W_{ij} N_j N_i = -\kappa^A V_{AA} - \kappa^B V_{BB} - 2\xi V_{AB}.$$

Recall now that div $V = 0$ and $VN = 0$ to infer from (2.16) that

$$\partial_N W_{NN} = -\partial_N V_{NN}.$$

Thus (3.16) is satisfied.

*Step* 5. The proof is achieved by the classical lifting theorem in $H^4(\Omega)$ for the components of $\Psi$ in the local basis such that (3.17)–(3.19) are satisfied. □

If $E$ is a symmetric matrix decomposed as (2.15), we denote by $E_T$ the tangential part of $E$ with components

$$(E_T)_{ij} := E_{AA}\tau_i^A\tau_j^A + E_{BB}\tau_i^B\tau_j^B + E_{AB}(\tau_i^A\tau_j^B + \tau_j^A\tau_i^B). \tag{3.29}$$

LEMMA 3.11. *Let $\mathbb{E} \in H^{3/2}(\partial\Omega, \mathbb{S}^3)$, $\mathbb{G} \in H^{1/2}(\partial\Omega, \mathbb{S}^3)$. There exists $H \in H^2(\Omega, \mathbb{S}^3)$ such that*

$$\begin{cases} H = \mathbb{E} & \text{on } \partial\Omega, \\ (\partial_N H)_T = \mathbb{G}_T & \text{on } \partial\Omega, \\ \operatorname{div} H = 0 & \text{on } \partial\Omega. \end{cases}$$

*Proof.* By the lifting theorem in $H^2(\Omega)$, one constructs functions $H_{AA}, H_{AB}, H_{BB} \in H^2(\Omega)$ such that on $\partial\Omega$

$$H_{AA} = \mathbb{E}_{AA}, \quad H_{AB} = \mathbb{E}_{AB}, \quad H_{BB} = \mathbb{E}_{BB},$$

$$\partial_N H_{AA} = \mathbb{G}_{AA}, \quad \partial_N H_{AB} = \mathbb{G}_{AB}, \quad \partial_N H_{BB} = \mathbb{G}_{BB}.$$

By (2.16), the conditions div $H = 0$ and $H = \mathbb{E}$ on $\partial\Omega$ impose $\partial_N H_{NN}$, $\partial_N H_{NA}$, and $\partial_N H_{NB}$ on $\partial\Omega$. Then one constructs $H_{NN}$, $H_{NA}$, and $H_{NB}$ in $\Omega$ using again the lifting theorem in $H^2(\Omega)$. □

THEOREM 3.12. *Let $\mathbb{E} \in H^{3/2}(\partial\Omega, \mathbb{S}^3)$ with*

$$\int_{\partial\Omega} \mathbb{E} N dS(x) = 0,$$

*i.e., $\mathbb{E} \in \tilde{H}^{3/2}(\partial\Omega, \mathbb{S}^3)$, and $\mathbb{G} \in H^{1/2}(\partial\Omega, \mathbb{S}^3)$. There exists $E \in H^2(\Omega, \mathbb{S}^3)$ such that*

$$\begin{cases} E = \mathbb{E} & \text{on } \partial\Omega, \\ (\partial_N E)_T = \mathbb{G}_T & \text{on } \partial\Omega, \\ \operatorname{div} E = 0 & \text{in } \Omega. \end{cases}$$

*In addition, such a lifting can be obtained through a linear continuous operator*

$$\mathcal{L}_{\partial\Omega} : (\mathbb{E}, \mathbb{G}) \in \tilde{H}^{3/2}(\partial\Omega, \mathbb{S}^3) \times H^{1/2}(\partial\Omega, \mathbb{S}^3) \mapsto E \in H^2(\Omega, \mathbb{S}^3).$$

*Proof.* Let $H$ be the function defined in Lemma 3.11. We must construct $K = E - H \in H^2(\Omega, \mathbb{S}^3)$ satisfying

$$\begin{cases} K = 0 & \text{on } \partial\Omega, \\ (\partial_N K)_T = 0 & \text{on } \partial\Omega, \\ \operatorname{div} K = -\operatorname{div} H & \text{in } \Omega. \end{cases}$$

We have div $H \in H_0^1(\Omega, \mathbb{R}^3)$ and

$$\int_\Omega \operatorname{div} H dx = \int_{\partial\Omega} HNdS(x) = \int_{\partial\Omega} \mathbb{E}NdS(x) = 0.$$

Therefore Theorem 3.10 provides the desired $K$. Finally, the linearity and the continuity of the obtained lifting are easily checked at each step of its construction. □

**3.3. Beltrami decomposition.** The following result is again given for the sake of generality in $L^p(\Omega)$ with $1 < p < \infty$ but should be considered here for $p = 2$.

THEOREM 3.13 (Beltrami decomposition [15]). *Assume that $\Omega$ is simply connected. Let $p \in (1, +\infty)$ be a real number, and let $E \in L^p(\Omega, \mathbb{S}^3)$ be a symmetric tensor. Then, for any $U \in W^{1/p,p}(\partial\Omega)$, there exist a unique $u \in W^{1,p}(\Omega, \mathbb{R}^3)$ with $u = U$ on $\partial\Omega$ and a unique $F \in L^p(\Omega, \mathbb{S}^3)$ with $\operatorname{Curl} F \in L^p(\Omega, \mathbb{R}^{3\times 3})$, $\operatorname{inc} F \in L^p(\Omega, \mathbb{S}^3)$, $\operatorname{div} F = 0$, and $FN = 0$ on $\partial\Omega$ such that*

$$E = \nabla^S u + \operatorname{inc} F. \tag{3.30}$$

*We call $\nabla^S u$ the compatible part and $E^0 := \operatorname{inc} F$ the (solenoidal) incompatible part of the Beltrami decomposition.*

**3.4. Green formula.** Let $V$ be a vector field defined on $\partial\Omega$, and let $\tilde{V}$ be any extension of $V$ in $\Omega$ with appropriate regularity. The surface divergence of $V$ is defined on $\partial\Omega$ by

$$\operatorname{div}_{\mathrm{S}} V = \operatorname{div} \tilde{V} - (\partial_N \tilde{V}) \cdot N. \tag{3.31}$$

LEMMA 3.14 (see [12]). *If $V \in W^{1,1}(\partial\Omega, \mathbb{R}^3)$, then*

$$\int_{\partial\Omega} \operatorname{div}_{\mathrm{S}} V dS(x) = \int_{\partial\Omega} \kappa V \cdot N dS(x).$$

LEMMA 3.15. *For all $A, B \in \mathcal{C}^2(\overline{\Omega}, \mathbb{M}^3)$,*

$$\int_\Omega A \cdot \operatorname{Curl} B dx = \int_\Omega \operatorname{Curl} A \cdot B dx + \int_{\partial\Omega} (A \times N) \cdot B dS(x).$$

*Proof.* We have

$$\begin{aligned} \int_\Omega A \cdot \operatorname{Curl} B dx &= \int_\Omega \epsilon_{jkm} A_{ij} \partial_k B_{im} dx \\ &= -\int_\Omega \epsilon_{jkm} \partial_k A_{ij} B_{im} dx + \int_{\partial\Omega} \epsilon_{jkm} A_{ij} B_{im} N_k dS(x) \\ &= \int_\Omega (\operatorname{Curl} A)_{im} B_{im} dx + \int_{\partial\Omega} (A \times N)_{im} B_{im} dS(x). \qquad \square \end{aligned}$$

Denote $A^S = (A + A^t)/2$ the symmetric part of a tensor $A$.

THEOREM 3.16. *Suppose that $T \in \mathcal{C}^2(\overline{\Omega}, \mathbb{S}^3)$ and $\eta \in H^2(\Omega, \mathbb{S}^3)$. Then*

$$\begin{aligned} \int_\Omega T \cdot \operatorname{inc} \eta dx = &\int_\Omega \operatorname{inc} T \cdot \eta dx \\ &+ \int_{\partial\Omega} \mathcal{T}_1(T) \cdot \eta \, dS(x) + \int_{\partial\Omega} \mathcal{T}_0(T) \cdot \partial_N \eta dS(x) \end{aligned} \tag{3.32}$$

*with the trace operators defined as*

$$\begin{aligned} &\mathcal{T}_0(T) := (T \times N)^t \times N, \\ &\mathcal{T}_1(T) := \left( \operatorname{Curl} (T \times N)^t \right)^S + ((\partial_N + \kappa) T \times N)^t \times N + \left( \operatorname{Curl}^{\mathrm{t}} T \times N \right)^S. \end{aligned} \tag{3.33}$$

*Proof.* By density we can assume that $\eta$ is smooth. Lemma 3.15 yields

$$\int_\Omega T \cdot \operatorname{inc} \eta dx = \int_\Omega \operatorname{Curl}^{\mathrm{t}} T \cdot \operatorname{Curl} \eta dx + \int_{\partial\Omega} \operatorname{Curl} \eta \cdot (T \times N)^t dS(x).$$

From the definition of the cross product of two tensors and its trace we observe that

$$\operatorname{div}\,(\operatorname{tr}\,A\times B)=\operatorname{Curl}\,A\cdot B-\operatorname{Curl}\,B\cdot A.$$

As a consequence, setting $A=(T\times N)^t$ and $B=\eta$ in the above identity, one has

$$\begin{aligned}\int_\Omega T\cdot\operatorname{inc}\,\eta dx=\int_\Omega \operatorname{Curl}^{\rm t}\,T\cdot\operatorname{Curl}\,\eta dx+\int_{\partial\Omega}\eta\cdot\operatorname{Curl}\,(T\times N)^t dS(x)\\ -\int_{\partial\Omega}\operatorname{div}\,\left(\operatorname{tr}\,\left((T\times N)^t\times\eta\right)\right)dS(x).\end{aligned}$$

By definition of the surface divergence, this is rewritten as

$$\begin{aligned}\int_\Omega T\cdot\operatorname{inc}\,\eta dx=\int_\Omega \operatorname{Curl}^{\rm t}\,T\cdot\operatorname{Curl}\,\eta dx+\int_{\partial\Omega}\eta\cdot\operatorname{Curl}\,(T\times N)^t dS(x)\\ -\int_{\partial\Omega}\left[\operatorname{div}_S\,\left(\operatorname{tr}\,\left((T\times N)^t\times\eta\right)\right)+\partial_N\left(\operatorname{tr}\,\left((T\times N)^t\times\eta\right)\right)\cdot N\right]dS(x).\end{aligned}$$

A short calculation shows that for two tensors $A$, $B$,

$$\operatorname{tr}\,(A\times B)\cdot N=-(A\times N)\cdot B.$$

Using Lemma 3.14 we obtain

$$\begin{aligned}\int_\Omega T\cdot\operatorname{inc}\,\eta dx=\int_\Omega \operatorname{Curl}^{\rm t}\,T\cdot\operatorname{Curl}\,\eta dx+\int_{\partial\Omega}\eta\cdot\operatorname{Curl}\,(T\times N)^t dS(x)\\ +\int_{\partial\Omega}\kappa(T\times N)^t\times N\cdot\eta dS(x)+\int_{\partial\Omega}\partial_N\left(((T\times N)^t\times N)\cdot\eta\right)dS(x).\end{aligned}$$

Rearranging yields

$$\begin{aligned}\int_\Omega T\cdot\operatorname{inc}\,\eta dx=\int_\Omega \operatorname{Curl}^{\rm t}\,T\cdot\operatorname{Curl}\,\eta dx+\int_{\partial\Omega}(T\times N)^t\times N\cdot\partial_N\eta dS(x)\\ +\int_{\partial\Omega}\left(\operatorname{Curl}\,(T\times N)^t+((\partial_N+\kappa)T\times N)^t\times N\right)\cdot\eta dS(x).\end{aligned}\tag{3.34}$$

One concludes the proof using Lemma 3.15. □

*Remark* 3.17. By Remark 3.4, only $(\partial_N\eta)_T$ matters in the rightmost integral of (3.32).

*Remark* 3.18. For a symmetric tensor $A$ and vectors $u$ and $v$, one has $((A\times u)^t\times v)^t = (A\times v)^t\times u$. Indeed, we have componentwise

$$\left((A\times u)^t\times v\right)_{ip}=\epsilon_{pjm}\epsilon_{ikl}A_{jk}u_l v_m=\epsilon_{ikl}\epsilon_{pjm}A_{kj}v_m u_l=\epsilon_{ikl}(A\times v)^t_{pk}u_l=(A\times v)^t\times u)_{pi}.\tag{3.35}$$

LEMMA 3.19. *We have the alternative expressions*

$$\begin{aligned}\mathcal{T}_1(T)=-\sum_R\kappa^R(T\times\tau^R)^t\times\tau^R-\sum_R\xi(T\times\tau^R)^t\times\tau^{R^*}+((-\partial_N+\kappa)T\times N)^t\times N\\ -2\left(\sum_R(\partial_R T\times N)^t\times\tau^R\right)^S,\end{aligned}\tag{3.36}$$

(3.37)
$$\mathcal{T}_1(T) = \sum_R \kappa^R (T\times\tau^R)^t \times \tau^R + \sum_R \xi (T\times\tau^R)^t \times \tau^{R^*} - ((\partial_N+\kappa)T\times N)^t \times N - 2\sum_R (\partial_R + \gamma^R)\left((T\times N)^t\times\tau^R\right)^S.$$

*In addition it holds that*

$$\int_{\partial\Omega} \mathcal{T}_1(T) N dS(x) = 0. \tag{3.38}$$

*Proof.* We have

$$\begin{aligned}
\left(\text{Curl } (T\times N)^t\right)_{mn} &= -\epsilon_{ikm}\epsilon_{jln}\partial_l(N_k T_{ij}) \\
&= -\epsilon_{ikm}\epsilon_{jln}(\partial_l N_k T_{ij} + N_k \partial_l T_{ij}) \\
&= -\epsilon_{ikm}\epsilon_{jln}\left(\sum_R \tau_l^R \partial_R N_k T_{ij} + N_k\partial_l T_{ij}\right) \\
&= -\epsilon_{ikm}\epsilon_{jln}\left(\sum_R (\kappa^R\tau_l^R\tau_k^R + \xi\tau_l^R\tau_k^{R^*})T_{ij} + N_k\partial_l T_{ij}\right) \\
&= -\sum_R \kappa^R\left((T\times\tau^R)^t\times\tau^R\right)_{nm} - \sum_R \xi\left((T\times\tau^R)^t\times\tau^{R^*}\right)_{nm} \\
&\quad + \left(\text{Curl}^{\text{t}}\ T\times N\right)_{nm}.
\end{aligned}$$

Hence
(3.39)
$$\text{Curl } (T\times N)^t = -\sum_R \kappa^R (T\times\tau^R)^t\times\tau^R - \sum_R \xi(T\times\tau^R)^t\times\tau^{R^*} + \left(\text{Curl}^{\text{t}}\ T\times N\right)^t.$$

By Lemma 3.1 and (3.35) we obtain (3.36).

Denote $E^A = E - E^S = \frac{1}{2}(E-E^t)$. By (3.39) and Lemma 3.1 we have

$$\left(\text{Curl } (T\times N)^t\right)^A = -\left(\text{Curl}^{\text{t}}\ T\times N\right)^A = \left(\sum_R (\partial_R T\times\tau^R)^t\times N\right)^A.$$

Integrating against $N$ and using the Stokes formula, by which $\int_{\partial\Omega}$ Curl $F N dS(x) = 0$ for any tensor $F$, yields

$$\int_{\partial\Omega} \text{Curl}^{\text{t}}\ (T\times N)^t N dS(x) = \int_{\partial\Omega}\left(\sum_R(\partial_R T\times\tau^R)^t\times N\right)^t N dS(x).$$

Using (3.35) and reordering the mixed product entails

$$\int_{\partial\Omega} \text{Curl}^{\text{t}}\ (T\times N)^t N dS(x) = -\int_{\partial\Omega}\left(\sum_R(\partial_R T\times N)^t\times N\right)\tau^R dS(x). \tag{3.40}$$

From the Stokes formula we have
(3.41)
$$\int_{\partial\Omega}\mathcal{T}_1(T)NdS(x) = \frac{1}{2}\int_{\partial\Omega}\text{Curl}^{\text{t}}\ (T\times N)^t N dS(x) + \frac{1}{2}\int_{\partial\Omega}\left(\text{Curl}^{\text{t}}\ T\times N\right)^t N dS(x),$$

and by Lemma 3.1
(3.42)
$$\left(\mathrm{Curl}^{\mathrm{t}}\ T \times N\right)^t N = -\sum_R \left((\partial_R T \times \tau^R)^t \times N\right)^t N = \sum_R \left((\partial_R T \times N)^t \times N\right) \tau^R.$$

Combining (3.41), (3.40), and (3.42) entails (3.38).

Finally, (3.37) is derived from (3.36) using

$$(\partial_R T \times N)^t \times \tau^R = \partial_R \left((T \times N)^t \times \tau^R\right) - (T \times \partial_R N)^t \times \tau^R - (T \times N)^t \times \partial_R \tau^R. \qquad \square$$

**3.5. Gauge conditions.** Theorem 3.16 applies to arbitrary test functions $\eta \in H^2(\Omega, \mathbb{S}^3)$, but only solenoidal fields are considered in the targeted application. The implications as to the dual characterization of the boundary term $\mathcal{T}_1(T)$ are discussed below. We define the gauge set

$$\mathcal{G} := \left\{V \odot N, V \in \mathbb{R}^3\right\} \subset \mathcal{C}^\infty(\partial\Omega, \mathbb{S}^3)$$

with

$$V_1 \odot V_2 := \frac{1}{2}(V_1 V_2^t + V_2 V_1^t) \qquad \forall\, V_1, V_2 \in \mathbb{R}^3,$$

and the matrices

$$M = \int_{\partial\Omega} N N^t dS(x), \qquad P = (|\partial\Omega| \mathbb{I}_2 + M)^{-1}.$$

In what follows we will denote duality pairings by integrals for the sake of readability.

Lemma 3.20. *Let $\mathbb{E} \in H^{-3/2}(\partial\Omega, \mathbb{S}^3)$. Then the condition*

$$\int_{\partial\Omega} \mathbb{E} \cdot F dS(x) = 0 \qquad \forall F \in \tilde{H}^{3/2}(\partial\Omega, \mathbb{S}^3) \tag{3.43}$$

*holds true if and only if $\mathbb{E} \in \mathcal{G}$.*

*Proof.* Assume first that $\mathbb{E} \in \mathcal{G}$, i.e., $\mathbb{E} = V \odot N$ for some $V \in \mathbb{R}^3$. We have for all $F \in \tilde{H}^{3/2}(\partial\Omega, \mathbb{S}^3)$

$$\int_{\partial\Omega} \mathbb{E} \cdot F dS(x) = \int_{\partial\Omega} (V \odot N) \cdot F dS(x) = \int_{\partial\Omega} (FN) \cdot V dS(x) = 0.$$

Assume now that $\mathbb{E} \in H^{-3/2}(\partial\Omega, \mathbb{S}^3)$ satisfies (3.43). Let $F \in H^{3/2}(\partial\Omega, \mathbb{S}^3)$ be arbitrary, and define

$$\Phi = \int_{\partial\Omega} F N dS(x), \qquad \tilde{F} = F - 2(P\Phi) \odot N.$$

We have

$$\begin{aligned}
\int_{\partial\Omega} \tilde{F} N dS(x) &= \Phi - 2 \int_{\partial\Omega} ((P\Phi) \odot N)\, N dS(x) \\
&= \Phi - \int_{\partial\Omega} \left(P\Phi N^t N + N N^t P\Phi\right) dS(x) \\
&= \Phi - (P\Phi |\partial\Omega| + M P\Phi) \\
&= \Phi - (|\partial\Omega| \mathbb{I}_2 + M) P\Phi = 0.
\end{aligned}$$

This implies that $\tilde{F} \in \tilde{H}^{3/2}(\partial\Omega, \mathbb{S}^3)$. Therefore

$$\begin{aligned}0 &= \int_{\partial\Omega} \mathbb{E} \cdot \tilde{F} dS(x) \\ &= \int_{\partial\Omega} \mathbb{E} \cdot F dS(x) - 2 \int_{\partial\Omega} (P\Phi) \cdot (\mathbb{E}N) dS(x) \\ &= \int_{\partial\Omega} \mathbb{E} \cdot F dS(x) - 2P \left( \int_{\partial\Omega} FN dS(x) \right) \cdot \left( \int_{\partial\Omega} \mathbb{E}N dS(x) \right).\end{aligned}$$

Set $V = P \int_{\partial\Omega} \mathbb{E}N dS(x)$. We obtain

$$0 = \int_{\partial\Omega} \mathbb{E} \cdot F dS(x) - 2V \cdot \left( \int_{\partial\Omega} FN dS(x) \right) = \int_{\partial\Omega} \left( \mathbb{E} - NV^t - VN^t \right) \cdot F dS(x).$$

This being true for all $F \in H^{3/2}(\partial\Omega, \mathbb{S}^3)$, we infer $\mathbb{E} = NV^t + VN^t = 2V \odot N$. ∎

We denote by $(\tilde{H}^{3/2}(\partial\Omega, \mathbb{S}^3))'$ the dual space of $\tilde{H}^{3/2}(\partial\Omega, \mathbb{S}^3)$. The restriction operator

$$\mathcal{R} : H^{-3/2}(\partial\Omega, \mathbb{S}^3) \to (\tilde{H}^{3/2}(\partial\Omega, \mathbb{S}^3))'$$

is surjective by the Hahn–Banach theorem, and Lemma 3.20 says that $\ker \mathcal{R} = \mathcal{G}$. Therefore, the reduced map $\tilde{\mathcal{R}} : H^{-3/2}(\partial\Omega, \mathbb{S}^3)/\mathcal{G} \to (\tilde{H}^{3/2}(\partial\Omega, \mathbb{S}^3))'$ is an isomorphism. This permits us to identify the dual of $\tilde{H}^{3/2}(\partial\Omega, \mathbb{S}^3)$ with the quotient space $H^{-3/2}(\partial\Omega, \mathbb{S}^3)/\mathcal{G}$.

LEMMA 3.21. *Every $\mathbb{E} \in H^{-3/2}(\partial\Omega, \mathbb{S}^3)/\mathcal{G}$ admits a unique representative $\tilde{\mathbb{E}}$ such that*

$$\int_{\partial\Omega} \tilde{\mathbb{E}} N dS(x) = 0. \tag{3.44}$$

*It is given by*

$$\tilde{\mathbb{E}} = \mathbb{E} - 2 \left( P \int_{\partial\Omega} \mathbb{E}N dS(x) \right) \odot N. \tag{3.45}$$

*Proof.* Arguing as in Lemma 3.20, one obtains that the function $\tilde{\mathbb{E}}$ defined by (3.45) satisfies (3.44). For the uniqueness, one has to show that if $\tilde{\mathbb{E}} \in \mathcal{G}$ satisfies (3.44), then $\tilde{\mathbb{E}} = 0$. Thus, suppose that $\tilde{\mathbb{E}} = V \odot N$, $V \in \mathbb{R}^3$. We have

$$\int_{\partial\Omega} \tilde{\mathbb{E}} N dS(x) = \frac{1}{2} \int_{\partial\Omega} (VN^t N + NN^t V) dS(x) = \frac{1}{2} P^{-1} V,$$

whereby (3.44) implies $V = 0$ and subsequently $\tilde{\mathbb{E}} = 0$. ∎

With these elements at hand, we can now generalize Theorem 3.16 to arbitrary tensors $T \in H_{\mathrm{inc}}(\Omega, \mathbb{S}^3)$. First we remark that, by density,

$$\int_\Omega T \cdot \operatorname{inc} \eta dx = \int_\Omega \operatorname{inc} T \cdot \eta dx \tag{3.46}$$

for every $T \in H_{\mathrm{inc}}(\Omega, \mathbb{S}^3)$ and $\eta \in H^2(\Omega, \mathbb{R}^3)$ such that $\eta = (\partial_N \eta)_T = 0$ on $\partial\Omega$. Then, for every $T \in H_{\mathrm{inc}}(\Omega, \mathbb{S}^3)$, we define the traces $\mathcal{T}_0(T) \in H^{-1/2}(\partial\Omega, \mathbb{S}^3)$ and $\mathcal{T}_1(T) \in H^{-3/2}(\partial\Omega, \mathbb{S}^3)/\mathcal{G}$ by

$$\langle \mathcal{T}_0(T), \varphi_0 \rangle = \int_\Omega T \cdot \operatorname{inc} \eta_0 dx - \int_\Omega \operatorname{inc} T \cdot \eta_0 dx \qquad \forall \varphi_0 \in H^{1/2}(\partial\Omega, \mathbb{S}^3),$$

$$\langle \mathcal{T}_1(T), \varphi_1\rangle = \int_\Omega T \cdot \operatorname{inc} \eta_1 dx - \int_\Omega \operatorname{inc} T \cdot \eta_1 dx \qquad \forall \varphi_1 \in \tilde{H}^{3/2}(\partial\Omega, \mathbb{S}^3),$$

with $\eta_0 = \mathcal{L}_{\partial\Omega}(0, \varphi_0)$ and $\eta_1 = \mathcal{L}_{\partial\Omega}(\varphi_1, 0)$ (recall that $\mathcal{L}_{\partial\Omega}$ is the lifting operator defined in Theorem 3.12). These definitions are independent of the choice of the liftings by virtue of (3.46). In addition, by Lemma 3.21, the function $\mathcal{T}_1(T)$ satisfying (3.38) is unambiguously defined in this way. By linearity of $\mathcal{L}_{\partial\Omega}$, this extends formula (3.32) to any functions $T \in H_{\mathrm{inc}}(\Omega, \mathbb{S}^3)$ and $\eta \in \mathcal{H}(\Omega)$.

Because of the aforementioned gauge properties, Lemmas 3.20 and 3.21 are also crucial in order to derive strong formulations. This issue is examined in the next section.

**4. A boundary value problem for the incompatibility.** In this section we assume that $\Omega$ is simply connected.

**4.1. Governing equations.** Let $\alpha \in L^\infty(\Omega)$ with $\inf_\Omega \alpha > 0$, $\mathbb{G} \in L^2(\Omega, \mathbb{S}^3)$ with $\operatorname{div} \mathbb{G} = 0$ in the sense of distributions. Consider the strictly convex minimization problem

$$\min_{E \in \mathcal{H}_0(\Omega)} \int_\Omega \left( \frac{\alpha}{2} |\operatorname{inc} E|^2 - \mathbb{G} \cdot E \right) dx, \tag{4.1}$$

whose Euler–Lagrange equation is

$$\int_\Omega \alpha \operatorname{inc} E \cdot \operatorname{inc} F dx = \int_\Omega \mathbb{G} \cdot F dx \qquad \forall F \in \mathcal{H}_0(\Omega). \tag{4.2}$$

By Theorem 3.9 and the Lax–Milgram theorem, (4.2) admits a unique solution $E \in \mathcal{H}_0(\Omega)$.

Remark that by Theorem 3.12, (4.2) also admits a unique solution in $\mathcal{H}_{\mathbb{E},\mathbb{F}}(\Omega)$ and even in $\mathcal{H}_{\mathbb{E},\mathbb{F};\Gamma_0}(\Omega)$ (with the test function $F \in \mathcal{H}_{\Gamma_0}(\Omega)$). In fact, it suffices to consider Theorem 3.9 and the Lax–Milgram theorem with the unknown $E - \mathcal{L}_{\partial\Omega}(\mathbb{E}, \mathbb{F})$ which satisfies homogeneous boundary conditions. Note that by (3.7), the tangential components of $\mathbb{F}$ are permutations of the components of $(\partial_N \mathbb{F})_T$ of Theorem 3.12. It should also be remarked that by the Green formula (3.32) and arguing as in section 4.2, the solution $E \in \mathcal{H}_{\mathbb{E},\mathbb{F};\Gamma_0}(\Omega)$ satisfies the Neumann conditions $\mathcal{T}_0(\alpha \operatorname{inc} E) = \mathcal{T}_1(\alpha \operatorname{inc} E) = 0$ on $\partial\Omega \setminus \Gamma_0$.

If $F \in \mathcal{D}(\Omega, \mathbb{S}^3)$ is not solenoidal, the Beltrami decomposition gives $F = F^0 + \nabla^S w$ with $\operatorname{div} F^0 = 0$ and $w = 0$ on $\partial\Omega$, and subsequently

$$\begin{aligned} \int_\Omega \alpha \operatorname{inc} E \cdot \operatorname{inc} F dx &= \int_\Omega \alpha \operatorname{inc} E \cdot \operatorname{inc} F^0 dx = \int_\Omega \mathbb{G} \cdot F^0 dx \\ &= \int_\Omega \mathbb{G} \cdot F dx - \int_\Omega \mathbb{G} \cdot \nabla^S w dx = \int_\Omega \mathbb{G} \cdot F dx, \end{aligned} \tag{4.3}$$

since $\operatorname{div} \mathbb{G} = 0$. Thus, in view of (3.32) with $T = \alpha \operatorname{inc} E$, the strong form associated with (4.3) is

$$\begin{cases} \operatorname{inc} (\alpha \operatorname{inc} E) = \mathbb{G} & \text{in } \Omega, \\ \operatorname{div} E = 0 & \text{in } \Omega, \\ E = 0 & \text{on } \partial\Omega, \\ (\partial_N E \times N)^t \times N = 0 & \text{on } \partial\Omega. \end{cases} \tag{4.4}$$

Let us first focus on the specific case where $\alpha$ is constant. We have the following.

LEMMA 4.1. *For all $E$ symmetric and solenoidal, it holds that* $\operatorname{inc}(\operatorname{inc} E) = \Delta\Delta E$.

*Proof.* Componentwise, one computes

$$\begin{aligned}(\operatorname{inc}(\operatorname{inc} E))_{ij} &= \varepsilon_{ikl}\varepsilon_{jmn}\varepsilon_{lpq}\varepsilon_{nrs}\partial_k\partial_m\partial_p\partial_r E_{qs}\\ &= (\delta_{ip}\delta_{kq}-\delta_{iq}\delta_{kp})(\delta_{jr}\delta_{ms}-\delta_{js}\delta_{mr})\partial_k\partial_m\partial_p\partial_r E_{qs}\\ &= \partial_i\partial_j\partial_q\partial_s E_{qs} - \partial_k^2\partial_j\partial_s E_{is} - \partial_k^2\partial_r\partial_j E_{ij} + \partial_p^2\partial_r^2 E_{ij} = \Delta\Delta E_{ij},\end{aligned}$$

which gives the expected result. ∎

By (3.6) and (3.7), the expression $(\partial_N E\times N)^t\times N$ is a mere linear recombination of $\partial_N E\times N$, whereby these two expressions are equivalent. Therefore, it is not difficult to see [21] that (4.4) for $\alpha$ constant is equivalent to

$$\begin{cases}\Delta(\alpha\Delta E) = \mathbb{G} & \text{in} \quad \Omega,\\ E = 0 & \text{on} \quad \partial\Omega,\\ \operatorname{div} E = 0 & \text{on} \quad \partial\Omega,\\ \partial_N E\times N = 0 & \text{on} \quad \partial\Omega,\\ \partial_N \operatorname{div} E = 0 & \text{on} \quad \partial\Omega.\end{cases} \tag{4.5}$$

Moreover, one has the following result [21].

THEOREM 4.2. *The system* (4.5) *admits a unique strong solution $E\in H^4(\Omega,\mathbb{S}^3)$, which is also a solution of* (4.2) *and* (4.4).

We infer the following property, which enables us to reconstruct a solenoidal tensor field from its incompatibility.

LEMMA 4.3. *Let $T\in L^2(\Omega,\mathbb{S}^3)$ with* $\operatorname{div} T = 0$ *in the sense of distributions. There exists $S\in H^2(\Omega,\mathbb{S}^3)$ such that* $\operatorname{div} S = 0$ *and* $\operatorname{inc} S = T$.

*Proof.* It suffices to set $S = \operatorname{inc} E$ with $E$ a solution of (4.4) with $\alpha = 1$ and $\mathbb{G} = T$. Theorem 4.2 shows that $E\in H^4(\Omega,\mathbb{S}^3)$; hence $S\in H^2(\Omega,\mathbb{S}^3)$. ∎

**4.2. Transmission conditions.** Let $\omega\subset\subset\Omega$ with smooth boundary $\partial\omega$ and outward unit normal $N$. Suppose that

$$\alpha = \begin{cases}\alpha_0 & \text{in } \Omega\setminus\omega,\\ \alpha_1 & \text{in } \omega,\end{cases}$$

with $\alpha_0$, $\alpha_1$ two positive constants.

THEOREM 4.4. *Assume $E$ is a solution of* (4.2), *and denote* $T = \operatorname{inc} E$. *Denote $[\![A]\!]$ the jump of the quantity $A$ across $\partial\omega$, with the trace counted positively on the interior side of $\omega$. Then*

$$\operatorname{inc}(\alpha T) = \mathbb{G} \qquad \textit{in } \Omega\setminus\partial\omega, \tag{4.6}$$

$$[\![\mathcal{T}_1(\alpha T)]\!] = 0 \qquad \textit{on } \partial\omega, \tag{4.7}$$

$$[\![\mathcal{T}_0(\alpha T)]\!] = 0 \qquad \textit{on } \partial\omega, \tag{4.8}$$

$$[\![TN]\!] = 0 \qquad \textit{on } \partial\omega. \tag{4.9}$$

*Conversely, if $T\in H_{\mathrm{inc}}(\Omega\setminus\partial\omega,\mathbb{S}^3)$ satisfies*

$$\operatorname{div} T = 0 \quad \textit{in } \Omega\setminus\partial\omega,$$

*together with* (4.6)–(4.9), *then*

$$\text{div } T = 0 \quad \textit{and} \quad \text{inc } (\alpha T) = \mathbb{G} \quad \textit{in } \Omega$$

*in the sense of distributions. Moreover, there exists* $E \in H^2(\Omega, \mathbb{S}^3)$ *with* div $E = 0$ *such that* $T = \text{inc } E$.

*Proof.* Using (3.32) (in its generalized version; see the discussion in section 3.4), we have for all $F \in \mathcal{H}_0(\Omega)$

$$\begin{aligned}\int_\Omega \mathbb{G} \cdot F dx &= \int_\omega \alpha_1 T \cdot \text{inc } F dx + \int_{\Omega\setminus\overline{\omega}} \alpha_0 T \cdot \text{inc } F dx \\ &= \int_\omega \text{inc } (\alpha_1 T) \cdot F dx + \int_{\Omega\setminus\overline{\omega}} \text{inc } (\alpha_0 T) \cdot F dx \\ &\quad + \int_{\partial\omega} [\![\mathcal{T}_1(\alpha T)]\!] \cdot F dS(x) + \int_{\partial\omega} [\![\mathcal{T}_0(\alpha T)]\!] \cdot \partial_N F dS(x).\end{aligned}$$

Choosing $F$ with compact support in $\omega$ then in $\Omega \setminus \omega$ yields (4.6), as in (4.4). By Theorem 3.12 combined with Lemmas 3.19, 3.20, and 3.21, we infer the two transmission conditions (4.7) and (4.8). In addition, one has div $T = 0$ which reads in the weak form

$$\int_\Omega T \cdot \nabla F dx = 0 \qquad \forall F \in \mathcal{D}(\Omega, \mathbb{R}^3).$$

Integrating by parts yields (4.9).

The converse relies on the standard Green formula, Theorem 3.16, and Lemma 4.3. ∎

**5. Physical interpretation.** The aim of this section is to describe two physically motivated problems where our model fourth-order boundary value problem with the inc operator is considered. In the first example special emphasis is given to the two Dirichlet boundary conditions, whereas in the second example, the main concern is the first Neumann boundary condition. In both cases, providing two Dirichlet boundary conditions on (arbitrarily small, but nonflat) $\Gamma_0$ is mandatory to ensure uniqueness of the solution.

**The displacement in linear elasticity in the presence of dislocations.** Let us assume that the distribution of dislocations is known and is given by the smooth second-rank tensor $\Lambda$ satisfying a local conservation law expressed in the form div $\Lambda^t = 0$, meaning that the dislocation lines are closed or end at the boundary [14, 23]. Let $\Gamma_0$ be a subset of $\partial\Omega$ which is not everywhere flat and has nonzero $\mathcal{H}^2$-measure. Let $\mathbb{F} \in H^{1/2}(\partial\Omega; \mathbb{S}^3)$ such that $\mathbb{F}N = 0$. By Lemma 3.1, one rewrites (1.7) with the second Dirichlet boundary condition restricted to $\Gamma_0$ and nonhomogeneous as

$$\begin{cases} \text{inc inc } E = \text{Curl } \kappa \quad \text{in} \quad \Omega, \\ E = 0 \quad \text{on} \quad \partial\Omega, \\ \text{Curl}^{\text{t}} \, E \times N = -\mathbb{F} \quad \text{on} \quad \Gamma_0. \end{cases} \tag{5.1}$$

It is understood that natural (homogeneous Neumann) boundary conditions complement this system. It is first observed from (1.4) that Curl $\kappa$ is symmetric as soon as

$\Lambda^t$ is divergence-free, since its skewsymmetric part vanishes, as seen by the following computation:

$$\epsilon_{mnp}\epsilon_{nkl}\partial_k(\kappa)_{ml} = \epsilon_{mnp}\epsilon_{nkl}\partial_k(\Lambda)_{ml} - \frac{1}{2}\epsilon_{mnp}\epsilon_{mnk}\partial_k(\Lambda)_{qq} \tag{5.2}$$

$$= \partial_p(\Lambda)_{ll} - \partial_m(\Lambda)_{mp} - \frac{1}{2}\epsilon_{mnp}\epsilon_{mnk}\partial_k(\Lambda)_{qq} = 0, \tag{5.3}$$

where we have used the identity $\epsilon_{mnp}\epsilon_{mnk} = 2\delta_{pk}$. Moreover, the second boundary condition is on the Frank tensor, the physical meaning of which is alluded to in the introduction.

Recall (3.2) and define

$$\mathcal{H}_{\mathbb{F};\Gamma_0}(\Omega) := \{E \in \mathcal{H}_{0,\mathbb{F};\Gamma_0}(\Omega) : E = 0 \text{ on } \partial\Omega\}. \tag{5.4}$$

Therefore, by our existence result for the nonhomogeneous problem, the field $E$ is found as the solution of

$$\min_{E\in\mathcal{H}_{\mathbb{F};\Gamma_0}(\Omega)} \int_\Omega \left(\frac{1}{2}|\operatorname{inc} E|^2 - \operatorname{Curl}\kappa\cdot E\right) dx. \tag{5.5}$$

Let us denote $\epsilon^0 := \operatorname{inc} E$. Now, by (1.6), one infers that the displacement $u$ is the solution of

$$\begin{cases} -\operatorname{div}(\mathbb{A}\nabla^S u) = \lambda\nabla\operatorname{tr}\epsilon^0 & \text{in} \quad \Omega, \\ (\mathbb{A}\nabla^S u)N = g - \lambda\operatorname{tr}\epsilon^0 N & \text{on} \quad \partial\Omega. \end{cases} \tag{5.6}$$

In this equation, one identifies $\lambda\nabla\operatorname{tr}\epsilon^0$ as a dislocation-induced conservative force in the body, and $-\lambda\operatorname{tr}\epsilon^0 N$ as a dislocation-induced traction at the boundary.

Remark that by Theorem 3.16 one has $\mathcal{T}_0(\epsilon^0) = 0$ on $\partial\Omega\setminus\Gamma_0$; i.e., the tangential components of $\epsilon^0$ vanish. Obviously one can take $\Gamma_0 = \partial\Omega$ to recover the full pure Dirichlet problem. It should also be noted that by Remark 3.5, taking $\mathbb{F} = 0$ in the second Dirichlet condition implies that $(\operatorname{inc} E)N = \epsilon^0 N = 0$ on $\Gamma_0$.

To summarize, in this section we have given a meaning to the equation

$$\begin{cases} -\operatorname{div}(\mathbb{A}\nabla^S u) = f & \text{in} \quad \Omega, \\ (\mathbb{A}\nabla^S u)N = g & \text{on} \quad \partial\Omega, \end{cases} \tag{5.7}$$

where $u$ is the displacement field and $f$ is a conservative force (as the gravity), in the case where $-\operatorname{div}\sigma = 0$ (global equilibrium) and in the presence of dislocations, i.e., $\nabla^S u = \mathbb{A}^{-1}\sigma - \epsilon^0$ with $\operatorname{inc}\epsilon^0$ related to the density of dislocations. Thus, we have started with the strain as variable, as in the intrinsic models of elasticity, and then introduced the displacement as the solutions of PDEs which describe the static problem of an elastic body with dislocations.

**Elements of a thermodynamic model for crystal growth.** Assume that the elastic body with dislocations is embedded in an environment whose temperature field $T$ is known. Assume that the dislocation density and hence the contortion tensor satisfy a constitutive law of the type

$$\kappa(T) = \mathbb{K}_{\mathsf{eq}}\exp\left(\frac{H}{k_b T_0}\left(\frac{T_0}{T} - 1\right) + S\left(1 - \frac{T}{T_0}\right)\left(\frac{T_0}{T} - 1\right)\right), \tag{5.8}$$

where $k_b$ is the Boltzman constant, $\mathbb{K}_{\mathsf{eq}}$ is the equilibrium concentration at the reference temperature $T_0$, and $H$ and $S$ are the effective formation enthalpy and entropy, respectively. A law such as (5.8) has been used successfully for the numerical simulation of point defects in single crystals, as reported in [22]. Referring to the brief discussion in the introduction and to Theorem 3.16, we would like to consider the mixed problem

$$\begin{cases} \text{inc}\,(\mathbb{M}\,\text{inc}\,\epsilon^0) = \mathbb{G} \quad \text{in} \quad \Omega, \\ \epsilon^0 = 0 \quad \text{on} \quad \partial\Omega, \\ \text{Curl}^{\text{t}}\,\epsilon^0 \times N = -\mathbb{F} \quad \text{on} \quad \Gamma_0, \\ \mathcal{T}_0(\text{inc}\,\epsilon^0) = \mathbb{T} \quad \text{on} \quad \partial\Omega \setminus \Gamma_0, \end{cases} \tag{5.9}$$

whose solution exists and is unique in $\mathcal{H}_{\mathbb{F};\Gamma_0}(\Omega)$ as defined in (5.4). Our aim is to physically interpret the boundary conditions of (5.9) and in particular the Neumann condition. First, the first Dirichlet boundary condition means by Beltrami decomposition that

$$\sigma N = (\mathbb{A}\epsilon)N = (\mathbb{A}\nabla^S u)N = g$$

on $\partial\Omega$; i.e., pure traction is exerted, with $u$ interpreted as the displacement field. As for the second condition, taking $\mathbb{F} = 0$ implies that $(\text{inc}\,\epsilon^0)N = (\text{Curl}\,\kappa)N = 0$ on $\Gamma_0$. By (5.8) this is rewritten as $\kappa'(T_0)(\nabla T \times N) = 0$ with $T_0$ the temperature at $\Gamma_0$, which is a condition satisfied if and only if $T = T_0$ is a constant on $\Gamma_0$, that is, the temperature gradient is purely normal on $\Gamma_0$. As for the first Neumann boundary condition, one has $\mathcal{T}_0(\text{Curl}\,\kappa) = (\text{Curl}\,\kappa \times N)^t \times N = \mathbb{T}$. By the symmetry property of $\kappa$ (cf. (5.2)), by Lemma 3.1 and (3.35), one has

$$(\text{Curl}\,\kappa \times N)^t = (\text{Curl}^{\text{t}}\,\kappa \times N)^t = -(\partial_N \kappa \times N)^t \times N - \sum_R (\partial_R \kappa \times N)^t \times \tau^R.$$

By (5.8), this yields

$$(\text{Curl}\,\kappa \times N)^t = -\partial_N T(\kappa'(T) \times N)^t \times N - (\kappa'(T) \times N)^t \times \nabla_S T,$$

where $\nabla_S T = \sum_R \tau^R \partial_R T$ means the surface gradient. Thus

$$(\text{Curl}\,\kappa \times N)^t \times N = \partial_N T(\kappa'(T) \times N)^t + \big((\kappa'(T) \times N)^t \times N\big) \times \nabla_S T. \tag{5.10}$$

We have assumed that the temperature field is known and hence $\mathbb{T}$ must be given by the right-hand side of (5.10), which involves the normal and tangential gradients of $T$. Note that the tangential gradient may not be zero if one thinks of a physical experiment such as Czochralski growth of single crystals [16, 22]. Moreover, in this case one can take $\Gamma_0$ as the solidification interface, where on the one hand the temperature is constant (and equal to the solidification temperature $T_0$), and which on the other hand is nowhere flat (by superficial tension properties).

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